Spectral Analysis of Saddle–point Matrices from Optimal Control PDE Problems

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Optimization problems with constraints given in terms of Partial Differential Equations are a ubiquitous problem of the applied mathematics. We consider here the distributed optimal control for the Poisson equation, and focus on the sequences of saddle–point linear systems stemming from its Finite Element approximation

$$\mathcal{A}_N \mathbf{x} \equiv \left[\begin{array}{cc|c} M & 0 & K^T \\ 0 & \alpha M & -M \\ \hline K & -M & 0 \end{array}\right] \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} M\mathbf{y}_d \\ \mathbf{0} \\ \mathbf{h} \end{bmatrix} \equiv \mathbf{b}, \qquad \begin{array}{l} M, K \in \mathbb{R}^{n^2 \times n^2}, \\ \alpha > 0, \\ N = 3n^2. \end{array}$$

Our main objective is then devising an efficient solution strategy for them by proving that the matrix sequence $\{\mathcal{A}_N\}_N$ belongs to the class of *Generalized Locally Toeplitz* sequences in their most general block multilevel setting. This framework permits describing the spectrum of the matrix sequence in a compact form, i.e., it ensures the existence of a measurable matrix–valued function κ, called symbol, correlated with the sequence. The knowledge of κ enables the detailed study of the distribution and the localization of the spectrum of the sequence together with the behavior of its extremal and outlier eigenvalues, and thus of its conditioning. By exploiting these results, we propose a specific solution strategy based on a preconditioned Krylov method. The study of the complexity, and of the convergence speed of the proposed method is then carried out by means of the information acquired from the symbol of the preconditioned sequence.

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