GMRES from a Hilbert space perspective

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We systematically re-derive a general residual minimizing Krylov subspace method for the solution of the linear system

$$A\,x = b, \tag{1}$$

where $A \in \mathcal{L}(X, X^*)$ is a non-singular operator and $b \in X^*$, where X is a Hilbert space and X^* is its dual and we denote the inner product imbuing X with the norm $\|\cdot\|_X$ and Hilbert space properties by $(\cdot,\cdot)_X$. The duality between elements of X and X^* is denoted by $\langle x, \xi \rangle_{X,X^*}$ or $\langle \xi, x \rangle_{X^*,X}$.
Since A maps X into X^*, which we must think of as different spaces (the primal solution space and dual residual space), powers of A cannot be formed. Thus the notion of preconditioner $P \in \mathcal{L}(X, X^*)$ must be an operator with the same mapping properties as A so that one can form powers of $AP^{-1} \in \mathcal{L}(X)$ and $P^{-1}A \in \mathcal{L}(X^*)$, and left- and right-preconditioning in this setting are seen rather as choosing to generate a Krylov subspace in either the primal or dual spaces. The preconditioner may still be thought to improve spectral properties of the problem, but its use is necessary here, first and foremost, to even consider constructing Krylov subspaces. In addition, we also consider $M, W \in \mathcal{L}(X, X^*)$ inducing positive bilinear forms to be used with the minimization and orthogonalization routines, respectively.
Beginning from the principles set out the original paper of Saad and Schultz (minimize the norm of the residual at every iteration over a Krylov subspace), we work in this general setting to understand what a *canonical* version of GMRES should look like. The Hilbert space perspective requires us to think about the realization of these principles in novel ways, and it turns out that these concepts are indeed sufficient to mathematically define the method in a unique way. Furthermore, in considering the existence of short-term recurrences in this setting, we recover a result of Faber-Manteuffel, which was published in a follow-up to their most well-known paper on that topic.

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