Graded L-type schemes coupled with an adaptive FEM for solving subdiffusion equations

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In this presentation, we consider the following subdiffusion or time-fractional diffusion equation

$$\mathrm{D}_t^\alpha u(\mathbf{x},t) - \nabla \cdot (\kappa \nabla u(\mathbf{x},t)) = f(\mathbf{x},t), \qquad (\mathbf{x},t) \in \Omega \times (0,T],$$

where T is the final time, f is a given source term, $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain, $\kappa > \kappa_0 > 0$, and D_t^α denote the Caputo fractional derivative of order $\alpha \in (0,1)$ which is defined as $\mathrm{D}_t^\alpha u(\cdot,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_t(\cdot,s)}{(t-s)^\alpha} ds$. For the first time, Stynes et. al investigated that the typical solution of the subdiffusion equation has a weak regularity near the initial time. This feature of the typical solution u leads to order reduction of schemes during using temporal uniform meshes. To achieve the expected convergence order, there are some efforts considering graded meshes with more realistic assumptions on the solution u. Furthermore, the solution of such a problem sometimes does not have the expected regularity w.r.t. the spatial variables and we have some regions with rapid variation, such as boundary layers and shocks. In this respect, the finite element method (FEM) with spatial adaptive meshes is proposed which refines the mesh locally instead of using quite fine mesh on the whole domain Ω.
In this work, we aim to investigate two following cases.

- We first consider the subdiffusion equation with smooth data in space direction and propose high-order graded L-type formulae, i.e. L1-2 and L1-2-3 schemes, mixed by a FEM to deal with the singularity near $t = 0$. The proposed methods have the $(3-\alpha)$- and 3-order accuracy in temporal direction concerning the maximum norm. We analzse the graded L1-2 FEM and graded L1-2-3 FEM similar to what Ramezani et. al have presented for temporal uniform meshes.
- Beside the previous case, we proceed to solve the equation when its solution contains sharp gradients. In this case, we use the proposed graded L1-2 and L1-2-3 formulae mixed by an adaptive FEM to achieve the expected 2-order accuracy in the spatial direction as well as the $(3-\alpha)$- and 3-order accuracy in the temporal direction, respectively.

Keywords: Subdiffusion equation, graded L-type formulae, stability and convergence analyses, adaptive FEM.

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