Space-time finite element and multigrid methods for coupled hyperbolic-parabolic systems

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We study the numerical approximation by space-time finite element methods (STFEMs) and the preconditioning of the resulting algebraic problem by geometric multigrid methods for the multi-physics system

$$\rho\partial_t^2\boldsymbol{u} - \nabla\cdot(\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) + \alpha\nabla p = \rho\boldsymbol{f}\,, \tag{1a}$$

$$c_0\partial_t p + \alpha\nabla\cdot\partial_t\boldsymbol{u} - \nabla\cdot(\boldsymbol{K}\nabla p) = g\,, \tag{1b}$$

supplemented with initial and boundary conditions. Problem (1) couples hyperbolic elastodynamics with parabolic transport and models poro- and thermoelasticity. Applications of (1) can be found in several branches, including power/reservoir engineering, material sciences and biomechanics/-medicine.

STFEMs allow the natural construction of higher order approximation schemes to (1). They can achieve accurate results on computationally feasible grids. Maintaining the stability and inheriting most of the rich structure of the continuous problem becomes increasingly difficult. Geometric multigrid methods are known as the most efficient iterative methods for the solution of large linear systems arising from the discretization of partial differential equations. To exploit their potential, they need to be adapted to the STFEMs and the mixed hyperbolic- parabolic character of (1).

For its discretization, Eq. (1) is rewritten as a first-order system in time. Galerkin methods in space and time with inf-sup stable pairs of finite elements for the spatial approximation of the unknowns are employed. Optimal order error estimates of energy-type are proven. Superconvergence at the discrete time nodes is addressed further. GMRES iterations with geometric multigrid preconditioning are used for the solution of the algebraic system, involving in parallel all temporal degress of freedom of the respective subinterval. The performance properties of the parallel V-cycle geometric multigrid preconditioner are investigated carefully by numerical experiments.

References:

[1] M. Bause, U. Köcher, F. A. Radu, Convergence of a continuous Galerkin method for mixed hyperbolic-parabolic systems, IMA J. Numer. Anal., submitted (2022), pp. 1-28; arXiv:2201.12014.

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