FE Approximation of Dirichlet Control Problems Governed by the Stokes System

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Control of fluids governed by the Stokes or Navier-Stokes equations is a question of major interest that has undergone a great development in the last years. We look for a velocity $\mathbf{y}$ that minimizes the distance in $\mathbf{L}^2$ to a prescribed target $\mathbf{y}_d$. To obtain $\mathbf{y}$, we can only control $\mathbf{u}$, its trace on the boundary of the domain, i.e., we want to solve

$$\min_{\mathbf{u}\in\mathbf{U}} \frac{1}{2}\|\mathbf{y}_\mathbf{u} - \mathbf{y}_d\|^2_{\mathbf{L}^2(\Omega)} + \text{regularization term}$$

subject to

$$-\Delta \mathbf{y}_\mathbf{u} + \nabla p = 0 \text{ in } \Omega,\ \nabla\cdot\mathbf{y}_\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{y}_\mathbf{u} = \mathbf{u} \text{ on } \Gamma, \int_\Omega p = 0,$$

where $\mathbf{U} \subset \mathbf{L}^2(\Gamma)$.

After giving a precise notion of solution in the transposition sense of the Stokes problem with Dirichlet data in Sobolev spaces with negative exponent, that will allow us to obtain optimal regularity results on polygonal domains, we will investigate three different kinds of regularization terms: usual $\mathbf{L}^2$ regularization, energy-space regularization and tangential control. We will show that the optimal solutions satisfy different regularity properties and these properties will give an upper bound in the order of convergence we can obtain from finite-dimensional approximations.

Next, we will focus in the discretization of the tangential-control problem using hybridizable discontinuous Galerkin methods, HDG. To do this, a proper mixed formulation that includes the flux of the velocity $\mathbb{L} = \nabla\mathbf{y}$ must be used. This represents a problem, since, for Dirichlet data in $\mathbf{L}^2(\Gamma)$, this flux is not a function. Moreover, for stabilization, the trace in the interior faces of the elements must be used. Thanks to our regularity results we are able to show that the flux is a function.

We will introduce two HDG formulations. First, a non-pressure-robust formulation that is able to deal even with what we call the “low-regularity” case, i.e., $\mathbf{y} \in \mathbf{H}^s(\Omega)$ for $s > 1$ and the trace of the flux may not be well defined. Second, we will show a pressure-robust variant. In this case, we can only treat the “high-regularity” case: $\mathbf{y} \in \mathbf{H}^s(\Omega)$ for $s > 3/2$ and the trace of the flux is be well defined. At the price of a slightly higher computational cost, we can get, for the approximation of the velocity $\mathbf{y}$, error estimates which are independent of the size of the pressure.

References:

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