A high-order FEM for distributed-order subdiffusion equations

Reza Mokhtari[1] Mohadese Ramezani[2] Gundolf Haase[3]

We consider the following distributed-order time-fractional diffusion, subdiffusion, equation

$$\begin{cases} \mathfrak{D}_t^\omega u - \Delta u = f(\mathbf{x},t), & \forall (\mathbf{x},t) \in \Omega \times (0,T], \\ u(\cdot,t)|_{\partial\Omega} = 0, \quad t \in [0,T]; & u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{cases}$$

where $\Omega \in \mathbb{R}^2$ is a bounded Lipschitz domain and $\mathfrak{D}_t^\omega$ is the distributed-order fractional derivative, defined by

$$\mathfrak{D}_t^\omega u(\cdot,t) = \int_0^\beta \omega(\alpha)\, \mathcal{D}_t^\alpha u(\cdot,t)\,\mathrm{d}\alpha, \qquad 0 < \beta \le 1,$$

in which $\omega(\alpha) \ge 0$, $\int_0^\beta \omega(\alpha)\,\mathrm{d}\alpha = c_0 > 0$ and $\mathcal{D}_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$ which is defined as $\mathcal{D}_t^\alpha u(\cdot,t) = (1/\Gamma(1-\alpha)) \int_0^t (t-s)^{-\alpha}\, u_t(\cdot,s)\,\mathrm{d}s$. The time-fractional distributed-order derivative is in fact a continuous generalization of the single-order time-fractional one, i.e. setting the single Dirac-δ function as ω leads to the ordinary time-fractional derivative. Hence, an effective numerical method for the distributed-order subdiffusion equation is necessary to develop. Utilizing the proposed formula by Mokhtari and Mostajeran, called L1-2-3, to approximate the distributed-order derivative and applying a finite element method (FEM) for spatial discretization, we develop a high temporal order fully discrete scheme. Combining with the idea of analysing the L1-2-3 FEM for the subdiffusion problem extended by Ramezani et al., we establish stability and convergence analyses of the proposed scheme. Detailed analysis indicates that the proposed L1-2-3 FEM is unconditionally stable and convergent with the convergence order $O(\tau^{4-\beta}|\ln\tau|^{-1} + h^2 + h_\alpha^2)$, where τ, h, and h_α are the step size in time, space, and distributed-order, respectively. The important point to note here is the improving the temporal convergence rate compare with the investigated method by Huang et al. in solving the distributed-order subdiffusion equation. Finally, some numerical examples are presented to support the theoretical prediction.

References:

[1] C. Huang, H. Chen, and N. An, β-Robust superconvergent analysis of a finite element method for the distributed order time-fractional diffusion equation, Journal of Scientific Computing, 90(1), (2022) 1-20.

[2] R. Mokhtari, and F. Mostajeran, A high order formula to approximate the Caputo fractional derivative, Communications on Applied Mathematics and Computation, 2(1), (2020) 1-29.

[3] M. Ramezani, R. Mokhtari, and G. Haase, Analysis of stability and convergence for L-type formulas combined with a spatial finite element method for solving subdiffusion problems, Electronic Transactions on Numerical Analysis, 55, (2022) 568-584.

[1]Isfahan University of Technology, Department of Mathematical Sciences, Isfahan 84156-83111, Iran
mokhtari@iut.ac.ir

[2]Isfahan University of Technology, Department of Mathematical Sciences, Isfahan 84156-83111, Iran
mohadeseh.ramezani@math.iut.ac.ir

[3]Institute for Mathematics and Scientific Computing, University of Graz, Graz, Austria
gundolf.haase@uni-graz.at