Multigrid in H(curl) on Hybrid Tetrahedral Grids

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This work presents theory and algorithms relevant to the solution of Maxwell's equations as well as their implementation in the massively scalable finite element framework *HyTeG*. We focus on multigrid methods for the curl-curl-problem which arises from the time-harmonic formulation of Maxwell's equations:

$$\alpha\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} + \beta\boldsymbol{u} = \boldsymbol{f}\,, \tag{1}$$

where $\alpha, \beta > 0$. This problem is not elliptic, rendering standard multigrid smoothers ineffective.

We resort to finite element exterior calculus (FEEC) to explain our choice of discretization: linear Nédélec edge elements of the first kind. Furthermore, FEEC directly leads to the Hodge decomposition, which is pivotal for the design of effective multigrid smoothers in $\boldsymbol{H}(\mathbf{curl})$. These were pioneered by Hiptmair, who proposed smoothing both in the Nédélec finite element space and the space of scalar potentials approximated by piecewise Lagrangian polynomials.

Novel is our implementation of the Nédélec space and associated grid transfer operators from/to the space of piecewise Lagrangian polynomials in *HyTeG*. The code makes use of code generation techniques to go from a mathematical description of operators and bilinear forms to efficient compute kernels automatically. This enables us to perform certain optimizations like common subexpression elimination on the symbolic level.

HyTeG is a finite element framework designed for massively parallel compute architectures. It supersedes the HHG framework which was already capable of solving systems with 10^{13} unknowns. The key building block to achieve these impressive results is a matrix-free implementation of geometric multigrid on hybrid tetrahedral grids. Using successive uniform refinement of a coarse mesh, we obtain a hierarchy of nested grids for multigrid and fast, indirection-free code. At the same time, the flexibility of unstructured topology is recovered by transforming the grid on all refinement levels to the actual domain yielding a curvilinear mesh.

References:

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