Solving Elliptic Partial Differential Equations on Metric Graphs using Multigrid Methods

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The question of investigating partial differential equations (PDEs) on graphs arises in the context of an interdisciplinary research project of the prediction of protein propagation in the brain network of Alzheimer's Disease patients [3]. Graphs allow modelling the structure of the brain network, while PDEs, particularly diffusion equations, describe the protein propagation. As a first attempt to approximate such PDEs on graphs, we focus on second order elliptic PDEs and multigrid methods for the solution of systems arising from a finite element discretization.
In order to formulate PDEs on graphs, we explain the network structure with the help of metric graphs. Metric graphs use an edgewise parameterization of the graph such that differential operators can be defined on graphs. Additionally, we require Neumann-Kirchhoff Conditions, a kind of flow conservation property, on all vertices.
We discretize the metric graph with a finite element method, as described in [1]. The discretization of the metric graph can be interpreted as an extended graph with additional vertices. We then choose a hat function basis on the extended graph, resulting in a characterisation of the weak formulation of the PDE as as

$$\begin{pmatrix} \mathbf{H}_{\mathcal{E}\mathcal{E}} & \mathbf{H}_{\mathcal{E}\mathcal{V}} \\ \mathbf{H}_{\mathcal{V}\mathcal{E}} & \mathbf{H}_{\mathcal{V}\mathcal{V}} \end{pmatrix} \mathbf{u} = \mathbf{f},$$

where $\mathbf{u}$ is the coefficient vector of the solution of the PDE written in its basis.
Each submatrix $\mathbf{H}_{\mathcal{E}\mathcal{E}}, \mathbf{H}_{\mathcal{E}\mathcal{V}}$, and $\mathbf{H}_{\mathcal{V}\mathcal{V}}$ corresponds to different adjacency of hat functions on the extended graph. Thus their size increase with more discretization points on each of the edges. This is especially important for the matrix $\mathbf{H}_{\mathcal{E}\mathcal{E}}$, because it is a block- diagonal matrix, with each blocks size proportional to the number of discretization points. Consequently, a fine discretization leads to a large system of equations and high computational cost.
We use a multigrid method to find the solution of the system of equations, making necessary adjustments for intergrid operators on graphs.
We show numerical results of the convergence rate of the multigrid method on Test problems.

References:

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