An anisotropic error estimate for SUPG space-time finite element methods

Ioannis Toulopoulos[1]

In this talk, we present SUPG space-time FE discretizations on anisotropic meshes for the following initial/boundary - value problem

$$u_t - \operatorname{div}(\varepsilon \nabla_x u) + \boldsymbol{\beta} \cdot \nabla_x u + ru = f \qquad \text{in } Q_T = (0,T] \times \Omega \tag{1a}$$

$$u = u_\Sigma = 0 \qquad \text{on } \Sigma := \partial\Omega \times [0,T], \tag{1b}$$

$$u(x,0) = u_0(x) \qquad \text{on } \Sigma_0 := \Omega \times \{0\}, \tag{1c}$$

where Ω is a bounded cuboid domain in $\mathbb{R}^{d_x}$, with $d_x = 1,2,3, T > 0$ a fixed time, $\nabla_x u$ is the spatial gradient of u, $\varepsilon > 0,\ r > 0$ are the diffusion and reaction coefficients, and $\boldsymbol{\beta} := (\beta_x, \beta_y, \beta_z)$ is a constant vector. The main idea of the proposed space-time scheme is to consider the temporal variable t as another spatial variable, let's say, x_{d_x+1}, and to consider u_t as a convection term in the direction x_{d_x+1}. In view of this, we discretize (**??**) in a unified way in the whole Q_T by applying SUPG finite element methodologies.
The main objective of this talk is to present a posteriori error bounds for the method, evaluated in the same SUPG norm used in the discretization error analysis. The corresponding error estimators are constructed using standard residual terms, including element-wise residuals and interface residuals arising from jumps in the normal fluxes across mesh faces. The development of these estimators relies on suitable anisotropic interpolation estimates, formulated for tensor-product finite-dimensional spaces on rectangular meshes. Notably, these estimates do not require restrictive assumptions on mesh stretching ratios or coordinate alignment. The proposed estimators yield an upper bound on the error measured in the SUPG norm. This is accomplished by analyzing the coercivity properties of the stabilized SUPG bilinear form with respect to that norm. In the final part of the talk, a series of numerical tests are presented to evaluate the properties and effectiveness of the proposed error estimators.
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[1]Department of Informatics, University of Western Macedonia, GREECE
itoulopoulos@uowm.gr