Nested Stochastic and Geometric Multigrid for Matrix-Free Space-Time Stochastic Galerkin Systems

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Stochastic Galerkin finite element methods can exploit smooth dependence on moderate-dimensional random inputs, but stochastic enrichment produces large and strongly coupled algebraic systems. We consider random parabolic problems discretized by conforming finite elements in space, discontinuous Galerkin methods in time, and Hermite generalized polynomial chaos in the stochastic variables. The resulting slab systems are applied matrix-free and solved by right-preconditioned FGMRES.

The main focus is a nested stochastic multigrid-geometric multigrid (SMG-GMG) preconditioner and its comparison with a block-Jacobi-GMG (BJ-GMG) baseline. BJ-GMG treats the diagonal stochastic blocks independently using deterministic space-time geometric multigrid, leaving the off-diagonal stochastic couplings to the outer Krylov iteration. SMG-GMG additionally resolves these couplings through a hierarchy in the total polynomial degree, forward and backward block Gauss-Seidel smoothing in stochastic mode space, and deterministic geometric multigrid for the individual block corrections.

For benchmarks with four Gaussian variables, (Q_3) finite elements in space, and (dG(3)) discretization in time, the average FGMRES iteration count for SMG-GMG remains nearly independent of the stochastic polynomial degree, increasing only from approximately (6.2) to (7.3). In contrast, the iteration count for BJ-GMG increases from approximately (9.6) to (30.7). At the higher stochastic degrees, SMG-GMG therefore requires about four times fewer outer iterations and exhibits no iteration growth under the investigated space-time refinements. Since an SMG-GMG cycle is more expensive than one BJ-GMG application, however, the resulting wall-time improvement is presently modest.

Finally, the solver framework is illustrated for contaminant transport driven by stationary Darcy flow with lognormal hydraulic conductivity. The experiments demonstrate improved robustness with respect to stochastic enrichment and identify the cost and parallelization of the stochastic smoother as the principal remaining challenges for scalable stochastic Galerkin simulations of more complex multiphysical systems such as porous-media problems.

References:

[1] M. Dawor, N. Margenberg, M. Bause, Stochastic Galerkin and Monte-Carlo methods for parabolic problems: Numerical performance of variational matrix-free approximations, preprint, 2026.

[2] M. Dawor, M. Bause, N. Margenberg, Matrix-free multigrid preconditioning for stochastic Galerkin space-time finite element discretizations, in preparation.

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