Singularity formation in nonlinear time-dependent PDEs

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Singularity formation in nonlinear partial differential equations

Many processes in natural sciences and applications are mathematically described by time-dependent PDEs (heat equation, wave equation, Schrödinger equation, Navier-Stokes equation, Einstein equations, ...)

Nonlinearities model self-reinforcing/focusing processes $\Rightarrow$ 'blowup' of solutions in finite time

Meaning?

- Limitation of the underlying modelling assumptions
- Physical system undergoes radical changes/formation of singularities
- Mathematically: change of solution concept $\Rightarrow$ continuation of solutions in some weak sense?

Singularity formation - Mathematical questions

- Criteria on initial data to predict break down of solutions?
- When/where/how fast do singularities form (blowup time/blowup point/blowup speed)?
- How do solutions look like close to the singularity?
- Continuation past the blowup?
- Behavior of generic solutions $\Rightarrow$ Universality?

Similar mechanisms seem to play a role in very different types of PDEs

Aim of this course

- Give a basic introduction into the topic
- Show classical methods that shed light on some of the above questions
- Make links to current fields of research

Remark: Singularity formation in nonlinear PDEs is a large and active area of research $\Rightarrow$ only a few aspects can be considered here!

Blowup in nonlinear ODEs

- Example 1: For $p > 1$, $p \in \mathbb{N}$

$$u'(t) = u(t)^p$$

has the blowup solution

$$u(t) = (T - t)^{-\frac{1}{p-1}} \kappa_p, \quad \kappa_p = \left(\tfrac{1}{p-1}\right)^{\frac{1}{p-1}}$$

- Example 2: For $p > 1$, $p \in \mathbb{N}$

$$u''(t) = u(t)^p$$

has the blowup solution

$$u(t) = (T - t)^{-\frac{2}{p-1}} c_p, \quad c_p = \left[\tfrac{2(p+1)}{(p-1)^2}\right]^{\frac{1}{p-1}}$$

- In contrast to *focusing* nonlinearities, *defocusing* nonlinearities behave better

$$u''(t) = -u(t)^p$$

for odd $p > 1 \Rightarrow$ no finite-time blowup

Nonlinear PDEs - model problems

Nonlinear wave equation on $\mathbb{R}^d$

$$\begin{aligned} \partial_t^2 u(t,x) - \Delta u(t,x) &= u(t,x)^p \\ u(0,x) = f(x), \partial_t u(0,x) &= g(x) \end{aligned}$$

Nonlinear heat equation on $\mathbb{R}^d$

$$\begin{aligned} \partial_t u(t,x) - \Delta u(t,x) &= u(t,x)^p \\ u(0,x) &= u_0(x) \end{aligned}$$

ODE blowup $\Rightarrow$ explicit example for finite-time blowup

What are the conditions on the initial data to ensure

- local existence of solutions for $t \in [0, T)$ and some $T > 0$?
- global existence or all $t > 0$, respectively finite time blowup of solutions?

The wave equation

Nonlinear wave equation on $\mathbb{R}^d$

$$\partial_t^2 u(t,x) - \Delta u(t,x) = \pm u(t,x)^p$$
$$u(0,x) = f(x), \partial_t u(0,x) = g(x)$$

for $(t,x) \in I \times \mathbb{R}^d$, $I \subset \mathbb{R}$ an interval containing 0

The wave equation

Linear wave equation on $\mathbb{R}^d$

$$\partial_t^2 u(t,x) - \Delta u(t,x) = F(t,x)$$
$$u(0,x) = f(x), \partial_t u(0,x) = g(x)$$

for $(t,x) \in I \times \mathbb{R}^d$, $I \subset \mathbb{R}$ an interval containing 0

The wave equation

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for $(t,x) \in I \times \mathbb{R}^d$, $I \subset \mathbb{R}$ an interval containing 0

- Free energy

$$E(u)(t) = \int_{\mathbb{R}^d} \left(|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 \right) dx$$

Basic energy estimate

$$\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)} + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^d)}$$
$$\lesssim \|\nabla f\|_{L^2(\mathbb{R}^d)} + \|g\|_{L^2(\mathbb{R}^d)} + \int_0^t \|F(s,\cdot)\|_{L^2(\mathbb{R}^d)} ds.$$

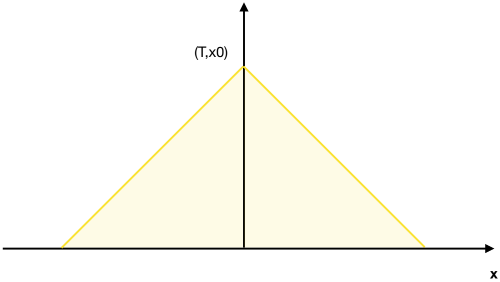
Representation of solutions - Properties

- Explicit solution $d = 1$, $F = 0$, d'Alembert's formula:

$$u(t,x) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds.$$

- Higher space dimensions: Kirchhoff formula, method of descent
- Finite speed of propagation $\Rightarrow$ backward lightcone at (T, x_0), $T > 0$, $x_0 \in \mathbb{R}^d$

$$\mathcal{C}_{T,x_0} := \{(t,x) \in \mathbb{R}^d : |x - x_0| \leq T - t, t \in [0,T)\}$$



Fourier transform: For $f \in \mathcal{S}(\mathbb{R}^d)$ we define the Fourier transform by

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x)dx$$

and

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi} \hat{f}(\xi)d\xi.$$

Recall:

- $\|\hat{f}\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}$ and $\|\hat{f}\|^2_{L^2(\mathbb{R}^d)} \simeq \|f\|^2_{L^2(\mathbb{R}^2)}$
- Convolution: $(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy$, $\mathcal{F}(f \star g) = \hat{f}\hat{g}$
- Derivatives: $\mathcal{F}(\partial^\alpha f)(\xi) = i\xi^\alpha \hat{f}(\xi)$

Sobolev spaces $H^k(\mathbb{R}^d)$: Completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to

$$\|f\|^2_{H^k(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1+|\xi|^2)^k |\hat{f}(\xi)|^2 d\xi = \|(1+|\cdot|^2)^{\frac{k}{2}} \hat{f}\|^2_{L^2(\mathbb{R}^d)}, \quad k \in \mathbb{N}_0$$

- For $k > \frac{d}{2}$, we have
$$\|f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{H^k(\mathbb{R}^d)}$$

Proof: For $f \in \mathcal{S}(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} |\hat{f}(\xi)| d\xi &= \int_{\mathbb{R}^d} (1+|\xi|^2)^{-k/2}(1+|\xi|^2)^{k/2}|\hat{f}(\xi)| d\xi \\ &\leq \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^{-k} d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^k |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &\leq C_k \|f\|_{H^k(\mathbb{R}^d)}\end{aligned}$$

with $C_k = \int_{\mathbb{R}^d} (1+|\xi|^2)^{-k} d\xi < \infty$ for $k > \frac{d}{2}$

The linear wave equation - Fourier representation of solutions

Fourier transform with respect to the spatial variable $\Rightarrow$

$$(\partial_t^2 + |\xi|^2)\hat{u}(t,\xi) = \hat{F}(t,\xi)$$
$$\hat{u}(0,\xi) = \hat{f}(\xi), \quad \partial_t \hat{u}(0,\xi) = \hat{g}(\xi)$$

Fundamental system $\{\sin(|\xi|\cdot), \cos(|\xi|\cdot)\}$

$$\begin{aligned}\hat{u}(t,\xi) &= c_1(\xi)\cos(|\xi|t) + c_2(\xi)\sin(|\xi|t)\\ &\quad - \cos(|\xi|t)\int_0^t \frac{\sin(|\xi|s)}{|\xi|}\hat{h}(s,\xi)ds + \sin(|\xi|t)\int_0^t \frac{\cos(|\xi|s)}{|\xi|}\hat{h}(s,\xi)ds\end{aligned}$$

Fourier representation

$$\hat{u}(t,\xi) = \cos(|\xi|t)\hat{f}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\hat{g}(\xi) + \int_0^t \frac{\sin(|\xi|(t-s))}{|\xi|}\hat{F}(s,\xi)ds$$

Duhamel's formula

$$u(t,\cdot) = \cos(|\nabla|t)f + \frac{\sin(|\nabla|t)}{|\nabla|}g + \int_0^t \frac{\sin(|\nabla|(t-s))}{|\nabla|}F(s,\cdot)ds$$

$$\cos(|\nabla|t)f := \mathcal{F}^{-1}(\cos(|\xi|t)\hat{f}), \quad \frac{\sin(|\nabla|t)}{|\nabla|}f = \mathcal{F}^{-1}\Big(\frac{\sin(|\xi|t)}{|\xi|}\hat{f}\Big)$$

The linear wave equation - Energy estimates

H^k- bounds for wave propergators

$$\|\cos(|\nabla|t)f\|_{H^k(\mathbb{R}^d)} \lesssim \|f\|_{H^k(\mathbb{R}^d)}, \quad \left\|\frac{\sin(|\nabla|t)}{|\nabla|}g\right\|_{H^k(\mathbb{R}^d)} \lesssim (1+t)\|g\|_{H^{k-1}(\mathbb{R}^d)}$$

Energy estimates for the linear wave equation

$$\begin{aligned} &\|u(t,\cdot)\|_{H^k(\mathbb{R}^d)} + \|\partial_t u(t,\cdot)\|_{H^{k-1}(\mathbb{R}^d)} \\ &\qquad \leq C(1+t)\left(\|f\|_{H^k(\mathbb{R}^d)} + \|g\|_{H^{k-1}(\mathbb{R}^d)} + \int_0^t \|h(s,\cdot)\|_{H^{k-1}(\mathbb{R}^d)} ds\right) \end{aligned}$$

The linear wave equation - Localized energy estimates

Localized energy

$$E_u^{loc}(t) := \int_{\mathbb{B}^d_{T-t}(x_0)} |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 dx$$

Note that $\frac{d}{dt} E_u^{loc}(t) \leq 0$

Energy estimates (localized)

For $x_0 \in \mathbb{R}^d$, $T > 0$ fix, $k \in \mathbb{N}$ and $0 < t < T$,

$$\|u(t,\cdot)\|_{H^k(\mathbb{B}^d_{T-t}(x_0))} + \|\partial_t u(t,\cdot)\|_{H^{k-1}(\mathbb{B}^d_{T-t}(x_0))}$$

$$\lesssim \|f\|_{H^k(\mathbb{B}^d_T(x_0))} + \|g\|_{H^{k-1}(\mathbb{B}^d_T(x_0))} + \int_0^t \|h(s,\cdot)\|_{H^{k-1}(\mathbb{B}^d_{T-s}(x_0))} ds$$

The nonlinear wave equation

Nonlinear wave equation on $\mathbb{R}^d$

$$\partial_t^2 u(t,x) - \Delta u(t,x) = \pm u(t,x)^p$$
$$u(0,x) = f(x), \partial_t u(0,x) = g(x)$$

with $p > 1$ an odd integer, $(t,x) \in I \times \mathbb{R}^d$, $I \subset \mathbb{R}$, $0 \in I$.

- Sign of the nonlinearity: focusing (+)/defocusing (-)
- Conserved energy

$$E(u)(t) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 dx \mp \frac{1}{p+1}\int_{\mathbb{R}^d} |u(t,x)|^{p+1} dx$$

- Notion of *criticality*: Invariance under rescaling

$$u_\lambda(t,x) = \lambda^{-\frac{2}{p-1}} u(t/\lambda, x/\lambda), \quad \lambda > 0$$

- Energy critical case $p = \frac{d+2}{d-2} =: p_c$

$$E(u_\lambda)(t) = E(u)(t/\lambda)$$

The nonlinear wave equation - Local well-posedness at high regularities

Definition: Strong H^k-solution

Set $F_\pm(u) = \pm u^p$. We say that $u \in C(I, H^k(\mathbb{R}^d) \cap C^1(I, H^{k-1}(\mathbb{R}^d))$ is a strong H^k-solution if it satisfies for all $t \in I$

$$u(t) = \cos(|\nabla|t)f + \frac{\sin(|\nabla|t)}{|\nabla|}g + \int_0^t \frac{\sin(|\nabla|(t-s))}{|\nabla|}F_\pm(u(s))ds$$

Theorem

Let $k > \frac{d}{2}$ and suppose $f \in H^k(\mathbb{R}^d), g \in H^{k-1}(\mathbb{R}^d)$. There is a $T > 0$ (depending on the norm of the data) such that the initial value problem for the nonlinear wave equation has a unique strong H^k-solution

$$u \in C([0,T], H^k(\mathbb{R}^d) \cap C^1([0,T], H^{k-1}(\mathbb{R}^d))$$

Basic idea: Solution via contraction mapping principle

- Duhamel's formula for linear wave equation
- L^∞ embedding to control the nonlinearity

Proof $d = 3$, $k = 2$

- $X := C([0,T], H^2(\mathbb{R}^d)) \cap C^1([0,T], H^1(\mathbb{R}^3))$ with norm

$$\|u\|_X := \sup_{0 \le t \le T} \left(\|u(t)\|_{H^2(\mathbb{R}^3)} + \|\partial_t u(t)\|_{H^1((\mathbb{R}^3)} \right)$$

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- For $R > 0$, consider

$$X_R := \{u \in X : \|u\|_X \le R\}$$

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- For $R > 0$, consider

$$X_R := \{u \in X : \|u\|_X \le R\}$$

- For $u \in X$ define a map $u \mapsto K(u)$,

$$K(u)(t) := \cos(|\nabla|t)f + \frac{\sin(|\nabla|t)}{|\nabla|}g + \int_0^t \frac{\sin(|\nabla|(t-s))}{|\nabla|}F_\pm(u(s,\cdot))ds$$

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- We show $K : X_R \to X_R$ is a contraction $\Rightarrow$ apply Banach's fixed point theorem

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- We show $K : X_R \to X_R$ is a contraction $\Rightarrow$ apply Banach's fixed point theorem
- By definition $K(u)$ solves linear wave equation with rhs. $F_\pm(u)$

- Energy estimates

$$\|K(u)(t)\|_{H^2(\mathbb{R}^3)} + \|\partial_t K(u)(t)\|_{H^1((\mathbb{R}^3)} \lesssim (1+t)\left(E_0 + \int_0^t \|u(s)^p\|_{H^1((\mathbb{R}^3)} ds\right)$$

  where $E_0 := \|f\|_{H^2} + \|g\|_{H^1}$
- Estimates for the nonlinearity, $u \in H^2(\mathbb{R}^3) \Rightarrow u \in L^\infty(\mathbb{R}^3)$

$$\|u^p\|_{H^1(\mathbb{R}^3)} \leq C'\|u\|^p_{H^2(\mathbb{R}^3)}$$

- Then

$$\|K(u)(t)\|_{H^2(\mathbb{R}^3)} + \|\partial_t K(u)(t)\|_{H^1(\mathbb{R}^3)} \leq C(1+T)(E_0 + TC'R^p) \leq R$$

  for $R > 0$ sufficiently large and $T \sim E_0^{-(p-1)}$ sufficiently small.
- Show $K : X_R \to X_R$ is a contraction: Let $u, v \in X_R$, then

$$K(u)(t) - K(v)(t) = \int_0^t \frac{\sin(|\nabla|(t-s))}{|\nabla|}[F_\pm(u(s)) - F_\pm(v(s))]ds$$

- Energy estimates

$$\|K(u)(t) - K(v)(t)\|_{H^2(\mathbb{R}^3)} + \|\partial_t K(u)(t) - \partial_t K(v)(t)\|_{H^1(\mathbb{R}^3)}$$
$$\leq C(1+T)\int_0^t \|u^p(s) - v^p(s)\|_{H^1(\mathbb{R}^3)} ds$$

- Use $u^p - v^p = (u-v)\sum_{j=0}^{p-1} u^{p-1-j}v^j$ to get

$$\|u^p - v^p\|_{H^1(\mathbb{R}^3)} \lesssim \|u-v\|_{H^2(\mathbb{R}^3)}(\|u\|^{p-1}_{H^2(\mathbb{R}^3)} + \|v\|^{p-1}_{H^2(\mathbb{R}^3)})$$

- We obtain

$$\|K(u) - K(v)\|_X \leq C_0(1+T)TR^{p-1}\|u-v\|_X \leq \frac{1}{2}\|u-v\|_X$$

for sufficiently small $T \sim E_0^{-(p-1)}$.

- Banach fixed point theorem $\Rightarrow$ Existence of a unique solution $u \in X_R$

- Unconditional uniqueness: Let $u, \tilde{u} \in X$ be strong H^k-solutions on $[0, T]$, set $v := u - \tilde{u}$

$$\begin{aligned} &\|v(t)\|_{H^2(\mathbb{R}^3)} + \|\partial_t v(t)\|_{H^1(\mathbb{R}^3)} \\ &\qquad \lesssim (1+t) \int_0^t \|v(s)\|_{H^2(\mathbb{R}^3)} (\|u(s)\|^{p-1}_{H^2((\mathbb{R}^3)} + \|\tilde{u}(s)\|^{p-1}_{H^2(R^3)}) ds \\ &\qquad \lesssim \int_0^t \|v(s)\|_{H^2(\mathbb{R}^3)} + \|\partial_s v(s)\|_{H^1(\mathbb{R}^3)} ds \end{aligned}$$

  Gronwall inequality $\Rightarrow v = 0$

- Persistence of regularity (smooth data implies smooth solution)

  Smoothness of the nonlinearity: For any $k > \frac{3}{2}$

$$\|u^p\|_{H^k(\mathbb{R}^3)} \lesssim \|u\|^p_{H^k(\mathbb{R}^3)}$$

$$\begin{aligned} \|u(t)\|_{H^3(\mathbb{R}^3)} + \|\partial_t u(t)\|_{H^2(\mathbb{R}^3)} &\lesssim \|f\|_{H^3(\mathbb{R}^3)} + \|g\|_{H^2(\mathbb{R}^3)} + \int_0^t \|u(s)^p\|_{H^2(\mathbb{R}^d)} ds \\ &\lesssim \|f\|_{H^3(\mathbb{R}^3)} + \|g\|_{H^2(\mathbb{R}^3)} + T\|u\|^p_X \end{aligned}$$

- Use energy estimates in lightcones $\Rightarrow$ Finite speed of propagation holds for nonlinear problem

Maximal solution/Blowup criterion

There is a $0 < T_+ \leq \infty$ such that

- There is a strong H^2- solution on $I_+ = [0, T_+)$ which is the only one in I_+
- If $\tilde{u}$ is another solution on some $I \subset [0, \infty)$, then $I \subseteq I_+$
- If $T_+ < \infty$ then $\limsup_{t \to T_+} (\|u(t)\|_{H^2(\mathbb{R}^3)} + \|\partial_t u(t)\|_{H^1(\mathbb{R}^3)}) = \infty$
- If $T_+ < \infty$ then $\|u(t, \cdot)\|_{L^\infty([0,T_+) \times \mathbb{R}^3)} = \infty$

Remarks

- The same strategy works in all space dimension at regularity $s > \frac{d}{2}$
- Local existence for negative times $[0, T_+) \to (T_-, T_+)$
- $T_\pm = \infty$?

The defocusing wave equation

Can the conserved energy be used to obtain global existence?

$$\|(u(t),\partial_t u(t))\|_{\dot{H}^1\times L^2(\mathbb{R}^d)} := \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} + \|\partial_t u(t)\|_{L^2(\mathbb{R}^d)}$$

- Energy estimates

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}+\|\partial_t u(t)\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}+\|g\|_{L^2(\mathbb{R}^d)}+\int_0^t \|u(s)^p\|_{L^2(\mathbb{R}^3)} ds$$

  Main challenge: Control of the nonlinear term

- Special case: $d=3$, $p=3$ Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$

$$\|u(t)^3\|_{L^2(\mathbb{R}^3)} = \|u(t)\|^3_{L^6(\mathbb{R}^3)} \lesssim \|\nabla u\|^3_{L^2(\mathbb{R}^3)}$$

  Fixed point argument in

$$X := \{C([0,T],\dot{H}^1(\mathbb{R}^3)) \cap C^1([0,T],L^2(\mathbb{R}^3))\}$$

  $X_R \subset X$, $R \sim \|(f,g)\|_{\dot{H}^1\times L^2(\mathbb{R}^3)} \Rightarrow$ solution on $[0,T]$, $T \sim R^{-2}$

- Defocusing case

$$\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 \leq E_{(f,g)}$$

  for all $t>0 \Rightarrow$ Global existence of solutions!

Local/global existence of solutions in the energy space

Remarks

- More general: Control of the nonlinearity via Strichartz estimates $\Rightarrow$ Local existence in the energy space for $1 < p \leq p_c$
- Global existence for the defocusing wave equation for $1 < p \leq p_c$
- Energy space not suitable for supercritical problems $p > p_c$
- No globally (in time) controlled quantities at higher Sobolev regularities

Big open question: Global existence for the defocusing wave equation in the supercritical case $p > p_c$?

The focusing wave equation - Finite-time blowup

For the focusing wave equation, finite-time blowup solutions do exist

$$\partial_t^2 u(t,x) - \Delta u(t,x) = u(t,x)^p$$
$$u(0,x) = f(x), \partial_t u(0,x) = g(x)$$

ODE blowup

$$u_T(t,x) = (T-t)^{-\frac{2}{p-1}} c_p, \quad c_p = \left[\tfrac{2(p+1)}{(p-1)^2}\right]^{\frac{1}{p-1}}, \quad T > 0$$

- Define smooth initial data (f,g) such that

$$f(x) = u_T(0,x), \quad g(x) = \partial_t u_T(0,x) \quad \forall x \in \overline{\mathbb{B}^3_{2T}}$$

  and $f(x) = g(x) = 0$ for $|x| \geq 3T$.
- Finite speed of propagation $\Rightarrow$ the solution blows up at $t = T$ on $\mathbb{B}^3_T$

We use this example to show that for $p = 7$ the initial value problem is not locally well-posed in the energy space $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

- We have the explicit solution $u(t,x) = (1-t)^{-\frac{1}{3}} c_7$
- Define initial data (f,g) as above $\Rightarrow$ the corresponding solution blows up at $t = 1$
- Rescaling $u_\lambda(t,x) = \lambda^{-\frac{1}{3}} u(t/\lambda, x/\lambda)$. Then

$$(u_\lambda, \partial_t u_\lambda)|_{t=0} = (f_\lambda, g_\lambda) = (u^*_\lambda, \partial_t u^*_\lambda)|_{t=0} \quad \text{in } \mathbb{B}^3_{2\lambda}$$

  $\Rightarrow$ $u_\lambda(t,x)$ blows up at $t = \lambda$
- Define a sequence $(\lambda_j) \subset \mathbb{R}^+$ such that

$$\lim_{j\to\infty} \lambda_j = 0, \quad \sum_{j=0}^{\infty} \lambda_j^{1/6} < \infty$$

- Define $(f_{\lambda_j}, g_{\lambda_j})$,

$$\|\nabla f_{\lambda_j}\|_{L^2(\mathbb{R}^3)} \lesssim \lambda_j^{1/6} \|\nabla f\|_{L^2(\mathbb{R}^3)}$$

  and similar for g_{λ_j}.

The focusing nonlinear wave equation

- Choose $x_j \in \mathbb{R}^3$, $\lim_{j\to\infty} x_j$ exists and such that the supports of

$$\tilde{f}_j(x) := f_{\lambda_j}(x - x_j), \quad \tilde{g}_j(x) := g_{\lambda_j}(x - x_j)$$

  are mutually disjoint.
- Define

$$\tilde{f} = \sum_{j=0}^{\infty} \tilde{f}_j, \quad \tilde{g} = \sum_{j=0}^{\infty} \tilde{g}_j$$

  Then

$$\|\nabla \tilde{f}\|_{L^2(\mathbb{R}^3)} \lesssim \sum_{j=0}^{\infty} \lambda_j^{1/6} \|\nabla f\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^3)}$$

  and similar for $\tilde{g}$
- By taking

$$\tilde{f} = \sum_{j=N}^{\infty} \tilde{f}_j, \quad \tilde{g} = \sum_{j=N}^{\infty} \tilde{g}_j$$

  for some large $N \in \mathbb{N} \Rightarrow$ data arbitrarily small in $\dot{H}^1 \times L^2(\mathbb{R}^d)$

The focusing wave equation - Finite-time blowup

Levine (1974): Negative energy $\Rightarrow$ Finite-time blowup

Let (f, g) be smooth, compactly supported initial data with

$$E_0 := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2 + |g(x)|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |f(x)|^{p+1} dx < 0$$

Then the corresponding solution cannot exist for all times.

Proof: e.g. [Evans, Chapter 12]

- Define $I(t) := \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^3)}$ and $J(t) := I(t)^{-\alpha}$, $2 + 4\alpha = p + 1$
- Use energy conservation
- $E(0) < 0 \Rightarrow J(t)$ is a convex function for all $t \geq 0$
- Argue by contradiction

The nonlinear heat equation

Nonlinear heat equation on $\mathbb{R}^d$

$$\begin{aligned} \partial_t u(t,x) - \Delta u(t,x) &= u(t,x)^p \\ u(0,x) &= u_0(x) \end{aligned}$$

- Scale invariance $u \mapsto u_\lambda$

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0$$

- Energy

$$E(u)(t) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u(t,x)|^{p+1} dx$$

  Energy dissipation $\frac{d}{dt}E(u)(t) \leq 0$, $\forall T > 0$

- 'Energy critical' exponent for $d \geq 3$

$$p_c := \frac{d+2}{d-2}$$

- Blowup in finite time if $E(u)(0) < 0$

The linear heat equation

We consider the Cauchy problem

$$\begin{aligned} \partial_t u(t,x) - \Delta u(t,x) &= 0 \qquad x \in \mathbb{R}^d, t > 0 \\ u(0,x) &= u_0(x) \end{aligned}$$

- Solution via Fourier transform $\Rightarrow$

$$u(t,x) = [G_t * u_0](x) = \int_{\mathbb{R}^d} G_t(x-y)u_0(y)dy =: [S(t)u_0](x)$$

  with heat kernel $G_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$
- $G_t(x) > 0$, $\forall x \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} G_t(x)dx = 1$
- Semigroup property: $G_{s+t} = G_s * G_t \Rightarrow$

$$S(t+s)u_0 = S(t)S(s)u_0, \quad \forall t > s > 0$$

- Maximum principle: $u_0(x) \geq 0 \Rightarrow u(t,x) > 0, \forall t > 0,\ x \in \mathbb{R}^d$
- $u_0 \geq 0 \Rightarrow$

$$\lim_{t\to\infty} (4\pi t)^{\frac{d}{2}} u(t,x) = \|u_0\|_{L^1(\mathbb{R}^d)}$$

The nonlinear heat equation - a Fujita-type result

Solution concept: Classical solutions or solutions that satisfy for $t \in [0, T)$

$$u(t) = S(t)u_0 + \int_0^t S(t-s)u(s)^p ds \tag{1}$$

- Fujita exponent $1 < p_F < p_c$

$$p_F = 1 + \frac{2}{d}$$

Theorem (Blowup for $1 < p \leq p_F$)

Let $1 < p \leq p_F$, $u_0 \geq 0 \in L^1(\mathbb{R}^d)$. Then there is no non-negative global solution to Eq. (1).

Remark: Such critical exponents can also be found for the nonlinear wave equation ('Strauss' exponent)

Sketch of the proof for $1 < p < p_F$, see book [Quittner-Souplet 2007, Sec.18]

- Basic idea: Compare decay estimates for 'free' evolution $S(t)u_0$ implied by Eq. (1) with linear decay rate $t^{-\frac{d}{2}}$
- More precisely, from Eq. (1) we obtain that

$$[S(t)u_0](x) \lesssim t^{-\frac{1}{p-1}}$$

- This implies

$$(4\pi t)^{\frac{d}{2}}[S(t)u_0](x) \lesssim t^{\frac{d}{2}-\frac{1}{p-1}}$$

such that for $1 < p < p_F$

$$\lim_{t\to\infty}(4\pi t)^{\frac{n}{2}}[S(t)u_0](x) = 0$$

which contradicts

$$\lim_{t\to\infty}(4\pi t)^{\frac{n}{2}}[S(t)u_0](x) = \|u_0\|_{L^1(\mathbb{R}^d)}.$$

Self-similar blowup solutions

- Scale invariant problems $\Rightarrow$ self-similar solutions?
- We first consider this for the nonlinear wave equation

Self-similar blowup solutions

$$u_T(t,x) = (T-t)^{-\frac{2}{p-1}} U\left(\frac{|x|}{T-t}\right)$$

- Insert this ansatz into the nonlinear wave equation $\Rightarrow$ nonlinear ODE

$$(1-\rho^2)U''(\rho) + \left(\frac{2}{\rho} - \frac{2(p+1)}{p-1}\rho\right) U'(\rho) - \frac{2(p+1)}{(p-1)^2} U(\rho) = U(\rho)^p$$

- Finite speed of propagation $\Rightarrow$ look for solutions that are smooth at least in a backward lightcone, i.e., for all $\rho \in [0,1]$
- Trivial solution $U_0(\rho) = c_p$ for all $d \geq 1$.
- Non-trivial profiles exist in the subcritical and supercritical case (ODE methods, numerics)

The focusing cubic wave equation - Non-trivial self-similar blowup

[Glogić-S., arXiv preprint (2018)] *Explicit* example: $p = 3$, $d \geq 5$,

$$u_T^*(t,x) = (T-t)^{-1} U^* \left(\frac{|x|}{T-t} \right), \quad U^*(\rho) = \frac{2\sqrt{2(d-1)(d-4)}}{d-4+3\rho^2}$$

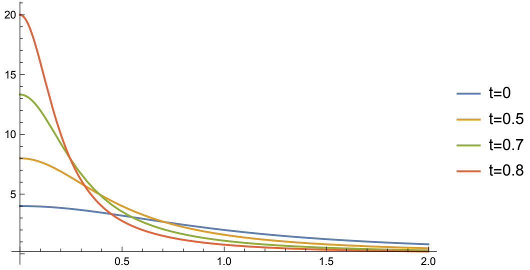


Figure: Blowup solution $u_1^*(t,r) = (1-t)^{-1} U^*(\frac{r}{1-t})$ for $d = 7$

The wave equation - Stable blowup behavior?

Q: Do self-similar solutions reflect properties of *generic* blowup solutions?

Numerical experiments [Bizoń-Chmaj-Tabor, Nonlinearity 17 (2004)]

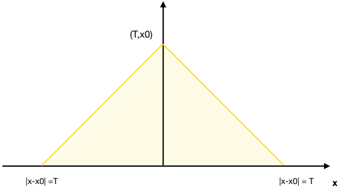
- ODE blowup describes behavior of generic blowup solutions locally around blowup point
- u_T^* appears at the threshold between finite-time blowup and global existence [Maliborski-Glogić-S., arXiv preprint (2019)]

Some analytic results

- $d = 1$: ODE blowup describes universal blowup behavior in backward lightcone of the blowup point [Merle-Zaag], J. Funct. Anal. 253 (2007)
- $d \geq 3$: ODE blowup is stable under small perturbations in backward lightcone of the blowup point [Donninger-S., Dyn. Partial Differ. Equ. 9 (2012), Trans. Amer. Math. Soc. 366 (2014)]
- Co-dimension one stability of u_T^* [Glogić-S., arXiv preprint (2018)]

- Reformulation of the problem using similarity coordinates

$$\xi = \frac{x - x_0}{T - t}, \quad \tau = -\log(T - t) + \log T$$



Analysis of self-similar blowup behavior

- Reformulation of the problem using similarity coordinates

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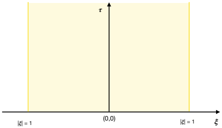

Analysis of self-similar blowup behavior

- Reformulation of the problem using similarity coordinates

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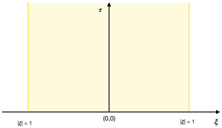

- Rescaled variable $\psi(\tau, \xi) = (T - t)^{\frac{2}{p-1}} u(t, x)$

$$\left(\partial_\tau^2 + \tfrac{p+3}{p-1}\partial_\tau + 2\xi^j \partial_{\xi^j}\partial_\tau - (\delta^{jk} - \xi^j \xi^k)\partial_{\xi^j}\partial_{\xi^k} + 2\tfrac{p+1}{p-1}\xi^j \partial_{\xi^j} + 2\tfrac{p+1}{(p-1)^2}\right)\psi = \psi^p$$

- Self-similar solutions that blowup at $(T, x_0) \Rightarrow$ static solution

The nonlinear heat equation - Self-similar blowup

Self-similar blowup solutions:

$$u(t,x) = (T-t)^{-\frac{1}{p-1}} f(\tfrac{x}{T-t}), \quad T > 0$$

The profiles w satisfy the elliptic equation

$$-\Delta w(y) + \frac{1}{2} y \cdot \nabla w(y) + \tfrac{1}{p-1} w(y) = w(y)^p \quad y \in \mathbb{R}^d$$

With $\sigma(y) = e^{-\frac{|y|^2}{4}}$ we write

$$-\nabla\Big(\sigma(y)\nabla w(y)\Big) = \sigma(y)\Big(w(y)^p - \tfrac{1}{p-1} w(y)\Big) \tag{2}$$

Constant solutions

$$w(y) = 0, \quad w(y) = \pm(1-p)^{-\frac{1}{p-1}}$$

In particular: The ODE blowup is a trivial self-similar solution

Theorem (Giga-Kohn, Comm. Pure Appl. Math. 38 (1985), no. 3)

For $1 < p \leq p_c$ there are no non-trivial solutions to Eq. (2).

<u>Sketch of the proof:</u> Multiply Eq. (2) with w and $|y|^2 w$ to obtain

$$\int_{\mathbb{R}^d} |\nabla w(y)|^2 \sigma(y)dy = \int_{\mathbb{R}^d} \left(|w(y)|^{p+1} - \tfrac{1}{p-1}|w(y)|^2\right)\sigma(y)dy \tag{3}$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} |y|^2|\nabla w(y)|^2 \sigma(y)dy = &\int_{\mathbb{R}^d} |y|^2|w(y)|^{p+1}\sigma(y)dy \\ &+ \int_{\mathbb{R}^d} \left(d|w(y)|^2 - \tfrac{p+1}{2(p-1)}|y|^2|w(y)|^2\right)\sigma(y)dy\end{aligned} \tag{4}$$

Multiplication by $y \cdot \nabla w$ implies

$$\begin{aligned}\int_{\mathbb{R}^d} \left(\tfrac{|y|^2}{4} + \tfrac{2-d}{2} \right) |\nabla w(y)|^2 \sigma(y) dy \\ = \int_{\mathbb{R}^d} (\tfrac{|y|^2}{2} - d) \left(\tfrac{1}{p+1} |w(y)|^{p+1} - \tfrac{1}{2(p-1)} |w(y)|^2 \right) \sigma(y) dy\end{aligned} \tag{5}$$

Then $2d\times$ (3) - (4) $+ 2(p+1)\times$ (5) yields

Pohozaev identity

$$\int_{\mathbb{R}^d} ((2-d)p + d + 2) |\nabla w(y)|^2 \sigma(y) dy + \frac{p-1}{2} \int_{\mathbb{R}^d} |y|^2 |\nabla w(y)|^2 \sigma(y) dy = 0$$

This implies that $w = const.$ for $d \leq 2$ or $d \geq 3$ and $1 < p \leq \frac{d+2}{d-2}$

For $p > p_c$, there exist radial non-trivial self-similar blowup solutions.

Example: $p = 2$, $7 \leq d \leq 15$,

$$u_T(x,t) = (T-t)^{-1} f\left(\frac{|x|}{\sqrt{T-t}}\right), \quad f(\rho) = \frac{24a}{(a+\rho^2)^2} + \frac{b}{a+\rho^2} \tag{6}$$

with constants

$$a = 2\left(10\sqrt{1+\tfrac{d}{2}} - d - 14\right), \quad b = 24\left(\sqrt{1+\tfrac{d}{2}} - 2\right).$$

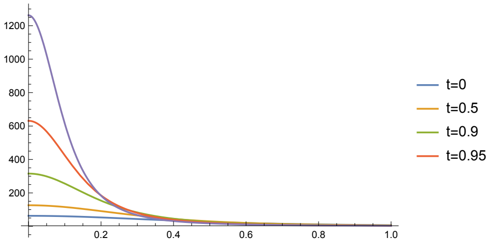

Figure: Blowup solution $u_T(t,r) = (1-t)^{-1} f\left(\frac{r}{1-t}\right)$ for $d = 7$, $T = 1$

Stable self-similar blowup in related models

Example 1 (co-rotational wave maps into $\mathbb{S}^3$)

$$\partial_t^2\psi(t,r) - \partial_r^2\psi((t,r) - \frac{2}{r}\partial_r\psi((t,r) + \frac{\sin(2\psi((t,r))}{r^2} = 0$$

Scale invariance $\psi \mapsto \psi_\lambda(t,r) := \psi(\lambda t, \lambda r)$, $\lambda > 0$

Self-similar blowup solution (gradient blowup)

$$\psi_T(t,r) = 2\arctan(\tfrac{r}{T-t})$$

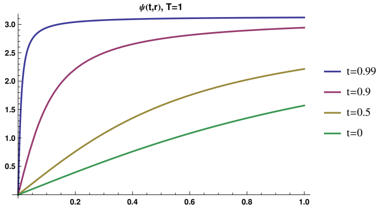

Figure: Blowup solution $\psi_1(t,r)$

Stable self-similar blowup in related models

Example 1 (co-rotational wave maps into $\mathbb{S}^3$)

$$\partial_t^2\psi(t,r) - \partial_r^2\psi((t,r) - \frac{2}{r}\partial_r\psi((t,r) + \frac{\sin(2\psi((t,r))}{r^2} = 0$$

Scale invariance $\psi \mapsto \psi_\lambda(t,r) := \psi(\lambda t, \lambda r)$, $\lambda > 0$

Extension past the blowup time [Biernat-Donninger-S., to appear in IMRN]

$$\psi_T^*(t,r) = 4\arctan\left(\frac{r}{T-t+\sqrt{(T-t)^2+r^2}}\right)$$

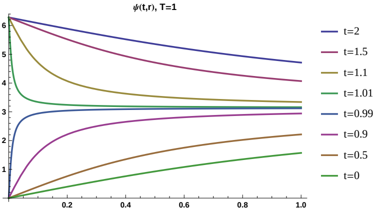

Figure: Blowup solution $\psi_1^*(t,r)$

Stable self-similar blowup in related models

Example 2 (Yang-Mills heat flow)

$$\partial_t u(r,t) - \partial_r^2 u(r,t) - \frac{d+1}{r}\partial_r u(r,t) = 3(d-2)u^2(r,t) - (d-2)r^2u^3(r,t)$$

Self-similar blowup solution for $d \geq 5$
[Weinkove, Calc. Var. PDE 19 (2004)]

$$u_T^*(r,t) = \tfrac{1}{T-t}W\left(\tfrac{r}{\sqrt{T-t}}\right), \quad W(\rho) = \frac{1}{a\rho^2+b}$$

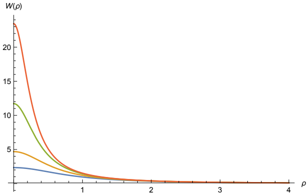

Figure: $u_T^*(r,t)$ in $d=5$ for $T=1$

Energy critical wave equation $p = p_c$

Explicit static solution

$$W(x) = \left(1 + \tfrac{|x|^2}{d(d-2)}\right)^{-(d-2)/2}.$$

[Kenig-Merle, Acta Math. 201 (2008)] For $E((f,g)) < E(W)$, if

- $\|\nabla f\|_{L^2} < \|\nabla W\|_{L^2} \Rightarrow$ Global existence.
- $\|\nabla f\|_{L^2} > \|\nabla W\|_{L^2} \Rightarrow$ Finite time blowup.

Type II blowup solutions

$$u(t,x) \sim \lambda(t)^{-\frac{2}{p-1}} W\left(\tfrac{x}{\lambda(t)}\right), \qquad \lambda(t) \to 0 \text{ as } t \to T$$

$d = 3$: [Krieger-Schlag-Tataru, Duke Math. J. 147 (2009)]

$d = 4$: [Hillairet-Raphaël, Anal. PDE 5 (2012)]

[. . .]

Further reading

Books/Lecture Notes

- L.C. Evans,**Partial Differential Equations (Second Edition)**
- C. Sogge, **Lectures on non-linear wave equations.**
- J. Shatah, M. Struwe, **Geometric wave equations**
- W. Strauss, **Nonlinear Wave Equations**
- T. Tao, **Nonlinear dispersive equations. Local and global analysis**
- P. Quittner, P. Souplet **Superlinear parabolic problems**

Thank you for your attention!