Basis properties of the Haar system in various function spaces, I.

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- Based on joint work with Gustavo Garrigós and Tino Ullrich

• A.S., T. Ullrich. Haar projection numbers and failure of unconditional convergence in Sobolev spaces. Mathematische Zeitschrift, 285 (2017), 91-119.

• ..., Lower bounds for Haar projections: Deterministic Examples. Constructive Approximation, 42 (2017), 227-242.

• G. Garrigós, A.S. and T. Ullrich. The Haar system as a Schauder basis in spaces of Hardy-Sobolev type. Journal of Fourier Analysis and Applications, 24(5) (2018), 1319-1339.

• ..., Basis properties of the Haar system in limiting Besov spaces. Preprint (arXiv).

• ..., The Haar system in Triebel-Lizorkin spaces: Endpoint cases. Preprint (arXiv).

The Haar system $\mathcal{H}$

Haar (1910): For $j \in \mathbb{N}_0$, $\mu \in \mathbb{Z}$ let $h_{j,\mu}$ be supported on $I_{j,\mu} = [2^{-j}\mu, 2^{-j}(\mu+1))$ and

$$h_{j,\mu}(x) = \begin{cases} 1 & \text{on the left half of } I_{j,\mu} \\ -1 & \text{on the right half of } I_{j,\mu} \end{cases}$$

- The *Haar frequency* of $h_{j,\mu}$ is 2^j.

- The functions $2^{j/2}h_{j,\mu}$, together with the functions $h_{-1,\mu} := 1_{[\mu,\mu+1)}$ form an ONB of $L^2(\mathbb{R})$.

- Let $\mathcal{H}$ be the collection of $h_{j,\mu}$, $j = -1, 0, 1, 2, \dots$, $\mu \in \mathbb{Z}$.

Haar system on $[0,1)$, or $\mathbb{T}$: Take only those Haar functions defined on $[0,1)$.

Haar system in d dimensions

- Intervals are replaced by cubes. For every dyadic cube we have $2^d - 1$ Haar functions.

$$\text{Let} \quad u^{(0)} = \mathbb{1}_{[0,1)}, \qquad u^{(1)} = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}.$$

For every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{0,1\}^d$ let

$$h^{(\varepsilon)}(x_1, \ldots, x_d) = u^{(\varepsilon_1)}(x_1) \cdots u^{(\varepsilon_d)}(x_d).$$

Finally, one sets

$$h_{j,\ell}^{(\varepsilon)}(x) = h^{(\varepsilon)}(2^j x - \ell), \quad j \in \mathbb{Z},\ \ell \in \mathbb{Z}^d,$$

The Haar system $\mathcal{H}_d$ is then given by

$$\mathcal{H}_d = \left\{ h_{0,\ell}^{(\vec{0})} \right\}_{\ell \in \mathbb{Z}^d} \cup \left\{ h_{j,\ell}^{(\varepsilon)} \mid j \in \mathbb{Z},\ \ell \in \mathbb{Z}^d,\ \varepsilon \in \{0,1\}^d \setminus \{\vec{0}\} \right\}.$$

Bases, I

Def. 1 Given a (quasi-)Banach space $\mathcal{X}$ of tempered distributions in $\mathbb{R}^d$ and an enumeration $\mathcal{U} = (u_1, u_2, \dots)$ of the Haar system $\mathcal{H}_d$, we say that $\mathcal{U}$ is a basic sequence on $\mathcal{X}$ if the orthogonal projections $P_n : \overline{\text{span}(\mathcal{U})} \to \text{span}\,(\{u_1, \dots, u_n\})$ are uniformly bounded.

• Only seemingly weaker: Any f in the closure of span$(\mathcal{U})$ can be expanded in a unique way as

$$f = \sum_n c_n(f) u_n$$

with convergence in $\mathcal{X}$. Then $c_n(f) = 2^{\text{freq}(u_n)} \langle f, u_n \rangle u_n$.

Def. 2. If $\mathcal{U}$ is a basic sequence on $\mathcal{X}$ and if span$(\mathcal{U})$ is dense in $\mathcal{X}$ then we say that $\mathcal{U}$ is a Schauder basis of $\mathcal{X}$.

Bases, II

Assume that span($\mathcal{U}$) is dense in $\mathcal{X}$ and suppose that $\mathcal{U}$ is a Schauder basis

Def. 3. $\mathcal{U}$ is an unconditional basis of $\mathcal{X}$ if for $f = \sum_n c_n u_n$ we have that

$$\sum_{n=1}^{\infty} c_{\varpi(n)} u_{\varpi(n)}$$

converges for every bijection $\varpi : \mathbb{N} \to \mathbb{N}$.

Equivalently:

- $\sum_{n=1}^{\infty} \pm c_n u_n$ converges for all choices of ± 1.
- $\sum_{n=1}^{\infty} \pm m(n) c_n u_n$ converges for all $m \in \ell^{\infty}(\mathbb{N})$.

Bases, III

Use the UBP:

• $\mathcal{U}$ is an unconditional basis if and only if the span of $\mathcal{U}$ is dense and if the projections to subspaces generated by finite subsets of $\mathcal{U}$ are uniformly bounded.

• For unconditional bases the multiplier problem is trivial:

$\mathcal{U}$ is an unconditional basis if and only if the multiplier transformation

$$f = \sum_n c_n(f) u_n \mapsto \sum_n m(n) c_n(f) u_n$$

is a bounded operator for all $m \in \ell^\infty(\mathbb{N})$.

Bases, IV

We say that $\mathcal{U}$ is an unconditional basic sequence on $\mathcal{X}$ if $\mathcal{U}$ is an unconditional basis on $\overline{\text{span}(\mathcal{U})}^{\mathcal{X}}$.

For the Haar system the following notion is also useful.
Def. $\mathcal{H}_d$ is a local basis of $\mathcal{X}$ if $f = \sum c_n(f)u_n$ converges for all compactly supported f.

The statements about projection operators remain true, but the operator norms depend on the choice of a compact set K in which all considered f are supported.

One can also define the notions of local unconditional basis, local basic sequence and local unconditional basic sequence.

The Haar basis in $L^p(\mathbb{R})$

Schauder (28): $\mathcal{H}$ (with the natural lexicographic order) is a *basis* of $L^p([0,1))$ when $1 \leq p < \infty$.

$$f = \mathbb{E}_0 f + \sum_{j=0}^{\infty} \sum_{\mu=0}^{2^j-1} 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu}$$

for $f \in L^p([0,1))$, with convergence in L^p.

• One works with conditional expectional operators $\mathbb{E}_N$ associated to dyadic intervals of length 2^{-N}.

• $\mathbb{E}_{N+1} - \mathbb{E}_N$ is the orthogonal projection to the space generated by the Haar functions with Haar frequency 2^N.

• Billard (1970's): $\mathcal{H}$ is a Schauder basis on the Hardy space $h^1(\mathbb{T})$.

The Haar basis in $L^p(\mathbb{R})$, $1 < p < \infty$

Marcinkiewicz (37): $\mathcal{H}$ is an *unconditional basis* of $L^p(\mathbb{R})$ when $1 < p < \infty$.

• For $f \in L^p$,

$$f = \sum_{j=-1}^{\infty} \sum_{\mu \in \mathbb{Z}} 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu}$$

with *unconditional* convergence in L^p.

Based on prior work of Paley, on square functions.

• Pełczynski (61): L^1 cannot be imbedded in a Banach space with an unconditional basis.

Function spaces on $\mathbb{R}^d$, I.

Sobolev spaces W_p^m, $1 \leq p \leq \infty$, $m \in \mathbb{N}$.

$$\|f\|_{W_p^m} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p$$

Bessel potential space L_s^p aka Sobolev space.

$$\|f\|_{L_s^p} = \|(I - \Delta)^{s/2} f\|_p$$

where $\mathcal{F}[(I - \Delta)^{s/2} f](\xi) = (1 + |\xi|^2)^{s/2} \widehat{f}(\xi)$.
Note that for $1 < p < \infty$ we have $W_s^p = L_s^p$ and the L_s^p interpolate with the complex method.

Since Haar functions are not smooth we are interested in these spaces for small s.

Function spaces on $\mathbb{R}^d$, II.

L^p Hölder classes $\Lambda(p, s) \equiv B^s_{p,\infty}$ For $0 < s < 1$, $1 \leq p \leq \infty$,

$$\|f\|_{B^s_{p,\infty}} = \|f\|_p + \sup_{h \neq 0} \frac{\|f(\cdot + h) - f\|_p}{|h|^s}.$$

Sobolev-Slobodecki spaces $B^s_{p,p}$. For $0 < s < 1$ let

$$\|f\|_{B^s_{p,p}} = \|f\|_p + \Big(\iint \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} dx\, dy \Big)^{1/p}$$

$B^s_{p,p}$ is also referred to as "Sobolev space of fractional order s".
But $B^s_{p,p} \neq L^p_s$ for $p \neq 2$.

Function spaces, III. The role of square functions

Consider $\{P_k\}_{k=0}^{\infty}$, an inhomogeneous dyadic frequency decomposition. Aka Littlewood-Paley decomposition.

Let $\phi_0 \in C_c^\infty((\mathbb{R}^d)^*)$, $\phi_0 = 1$ near 0.

$$\widehat{P_0 f}(\xi) = \phi_0(\xi)\widehat{f}(\xi),$$
$$\widehat{P_k f}(\xi) = (\phi_0(2^{-k}\xi) - \phi_0(2^{1-k}\xi))\widehat{f}(\xi), \quad k \geq 1.$$

• Localization to frequencies of size $\approx 2^k$.
Then, for $1 < p < \infty$

$$\|f\|_{L^p_s} = \Big\| \Big(\sum_{k=0}^{\infty} 2^{2ks} |P_k f|^2 \Big)^{1/2} \Big\|_p.$$

by standard singular integral theory (in a Hilbert-space setting).

Function spaces, IV. $B^s_{p,q}$, $F^s_{p,q}$

"Function spaces" as subspaces of tempered distributions via Fourier analytic definitions:

Besov-Nikolskij-Taibleson spaces, $0 < p, q \leq \infty$.

$$\|f\|_{B^s_{p,q}} = \left\|\{2^{ks} P_k f\}\right\|_{\ell^q(L^p)}$$

Triebel-Lizorkin spaces. $0 < p < \infty$, $0 < q \leq \infty$.

$$\|f\|_{F^s_{p,q}} = \left\|\{2^{ks} P_k f\}\right\|_{L^p(\ell^q)}$$

Note $F^s_{p,2} = L^p_s$, $1 < p < \infty$. Hardy-Sobolev H^s_p when $p > 0$.

There is an extension to $p = \infty$, so that $F^0_{\infty,2} = BMO$, using BMO-like norms in the general case (cf. the Chang-Wilson-Wolff theorem and Frazier-Jawerth definitions).

Function spaces, V. Peetre maximal functions

Motivated by the Hardy space results of Fefferman-Stein, Peetre (1975) introduced maximal functions on distributions with bounded Fourier theorems about maximal functions support. Assume f Schwartz and $\widehat{f}$ supported in a set of diameter 1.

Then

$$\sup_{x\in\mathbb{R}^d} \frac{|f(x+h)|}{(1+|h|)^{d/r}} \lesssim \left(M_{HL}[|f|^r]\right)^{1/r}.$$

Summary of proof: One proves first

$$\sup_{x\in\mathbb{R}^d} \frac{|\nabla f(x+h)|}{(1+|h|)^{d/r}} \lesssim \sup_{x\in\mathbb{R}^d} \frac{|f(x+h)|}{(1+|h|)^{d/r}}$$

and then relies on a mean value inequality

$$|g(x)| \leq c_1 \delta \sup_{B_\delta(x)} |\nabla g| + c_2 \Big(\mathrm{av}_{B_\delta(x)} |g|^r\Big)^{1/r}.$$

Function spaces, VI. Peetre maximal functions: Scaled and vector valued versions

• Assume that $\widehat{f_k} \in \mathcal{S}'$ is supported on a set of diameter R_k. Let

$$\mathcal{M}_{k,A} f_k(x) = \sup_{h \in \mathbb{R}^d} \frac{|f_k(x+h)|}{(1 + R_k|h|)^A}.$$

Then

$$\big\|\{\mathcal{M}_{k,A} f_k\}\big\|_{\ell^q(L^p)} \lesssim_A \big\|\{f_k\}\big\|_{\ell^q(L^p)}, \quad A > d/p.$$

$$\big\|\{\mathcal{M}_{k,A} f_k\}\big\|_{L^p(\ell^q)} \lesssim_A \big\|\{f_k\}\big\|_{L^p(\ell^q)}, \quad A > d/p, A > d/q.$$

One can use Fefferman-Stein vector-valued extension of the Hardy-Littlewood maximal theorem.

Is the additional condition on q needed?

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Is the additional condition on q needed? Yes, no matter what the R_k are. (Christ, S., PLMS 2006).

Questions for function spaces measuring smoothness

Consider Triebel-Lizorkin spaces $F^s_{p,q}$, Besov spaces $B^s_{p,q}$.

Q1: For which spaces is $\mathcal{H}_d$ a basic sequence?

Q2: For which spaces is $\mathcal{H}_d$ a Schauder basis?

Q3: For which spaces is $\mathcal{H}_d$ an unconditional basis?

Q4: Haar system on unit cube or on $\mathbb{R}^d$: Does it matter for the outcomes?

- Obvious necessary condition: The Haar functions must belong to the space (mostly $s < 1/p$).
- Other necessary conditions by duality (e.g. mostly $s > -1 + 1/p$ when $1 < p < \infty$).
- Interpolation gives additional restrictions for cases with $p \leq 1$.
- We often disregard the cases $p = \infty$ or $q = \infty$ (Schwartz functions are not dense).

Some references to prior work

• Triebel (73), (78): $\mathcal{H}_d$ is an (unconditional) basis on $B^s_{p,q}$ if

$$\max\{\frac{d}{p} - d,\ \frac{1}{p} - 1\} < s < \min\{\frac{1}{p}, 1\}.$$

Result is sharp up to endpoints. Secondary smoothness parameter q plays no role.
• Many more results on splines, wavelets in Besov spaces (Ciesielski, Figiel, Ropela, Meyer, Sickel, Bourdaud, Oswald).

2010: Triebel's monograph :
$\mathcal{H}_d$ is an unconditional basis on $F^s_{p,q}$ if

$$\max\{\frac{1}{p} - 1,\ \frac{1}{q} - 1,\ \frac{d}{p} - d, \frac{d}{q} - d\} < s < \min\{\frac{1}{p}, \frac{1}{q}, 1\}.$$

Q: Is the additional restriction on q necessary?

A recurring picture

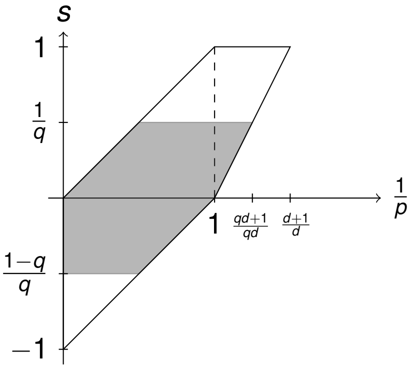

Figure: Parameter domains for the Haar system in $F^s_{p,q}$ spaces on $\mathbb{R}^d$, $1 < q < \infty$, here $q = 2$.

$\mathcal{H}_d$ is unconditional basis for $B^s_{p,q}$: interior of the entire domain.

More about this on Thursday.