Basis properties of the Haar system in various function spaces, I.

Andreas Seeger (University of Wisconsin-Madison)

Chemnitz Summer School on Applied Analysis 2019

- Based on joint work with Gustavo Garrigós and Tino Ullrich


Haar (1910): For \( j \in \mathbb{N}_0, \mu \in \mathbb{Z} \) let \( h_{j,\mu} \) be supported on \( I_{j,\mu} = [2^{-j}\mu, 2^{-j}(\mu + 1)) \) and

\[
h_{j,\mu}(x) = \begin{cases} 
1 & \text{on the left half of } I_{j,\mu} \\
-1 & \text{on the right half of } I_{j,\mu}
\end{cases}
\]

- The Haar frequency of \( h_{j,\mu} \) is \( 2^j \).
- The functions \( 2^j/2 h_{j,\mu} \), together with the functions \( h_{-1,\mu} := 1_{[\mu,\mu+1)} \) form an ONB of \( L^2(\mathbb{R}) \).
- Let \( \mathcal{H} \) be the collection of \( h_{j,\mu}, j = -1, 0, 1, 2, \ldots, \mu \in \mathbb{Z} \).

Haar system on \([0, 1)\), or \( \mathbb{T} \): Take only those Haar functions defined on \([0, 1)\).
Haar system in $d$ dimensions

- Intervals are replaced by cubes. For every dyadic cube we have $2^d - 1$ Haar functions.

Let $u^{(0)} = \mathbb{I}_{[0,1)}$, $u^{(1)} = \mathbb{I}_{[0,1/2)} - \mathbb{I}_{[1/2,1)}$.

For every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{0, 1\}^d$ let

$$h^{(\varepsilon)}(x_1, \ldots, x_d) = u^{(\varepsilon_1)}(x_1) \cdots u^{(\varepsilon_d)}(x_d).$$

Finally, one sets

$$h^{(\varepsilon)}_{j,\ell}(x) = h^{(\varepsilon)}(2^j x - \ell), \quad j \in \mathbb{Z}, \ \ell \in \mathbb{Z}^d,$$

The Haar system $\mathcal{H}_d$ is then given by

$$\mathcal{H}_d = \left\{ h^{(0)}_{0,\ell} \right\}_{\ell \in \mathbb{Z}^d} \cup \left\{ h^{(\varepsilon)}_{j,\ell} \mid j \in \mathbb{Z}, \ \ell \in \mathbb{Z}^d, \ \varepsilon \in \{0, 1\}^d \setminus \{\vec{0}\} \right\}.$$
Def. 1 Given a (quasi-)Banach space $\mathcal{X}$ of tempered distributions in $\mathbb{R}^d$ and an enumeration $\mathcal{U} = (u_1, u_2, \ldots)$ of the Haar system $\mathcal{H}_d$, we say that $\mathcal{U}$ is a basic sequence on $\mathcal{X}$ if the orthogonal projections $P_n : \text{span}(\mathcal{U}) \to \text{span} \{u_1, \ldots, u_n\}$ are uniformly bounded.

- Only seemingly weaker: Any $f$ in the closure of $\text{span}(\mathcal{U})$ can be expanded in a unique way as

$$f = \sum_n c_n(f) u_n$$

with convergence in $\mathcal{X}$. Then $c_n(f) = 2^{\text{freq}(u_n)} \langle f, u_n \rangle u_n$.

Def. 2. If $\mathcal{U}$ is a basic sequence on $\mathcal{X}$ and if $\text{span}(\mathcal{U})$ is dense in $\mathcal{X}$ then we say that $\mathcal{U}$ is a Schauder basis of $\mathcal{X}$. 
Assume that \( \text{span}(\mathcal{U}) \) is dense in \( X \) and suppose that \( \mathcal{U} \) is a Schauder basis

**Def. 3.** \( \mathcal{U} \) is an **unconditional basis** of \( X \) if for \( f = \sum_n c_n u_n \) we have that

\[
\sum_{n=1}^{\infty} c_{\varpi}(n) u_{\varpi}(n)
\]

converges for every bijection \( \varpi : \mathbb{N} \to \mathbb{N} \).

Equivalently:

- \( \sum_{n=1}^{\infty} \pm c_n u_n \) converges for all choices of \( \pm 1 \).
- \( \sum_{n=1}^{\infty} \pm m(n) c_n u_n \) converges for all \( m \in \ell^\infty(\mathbb{N}) \).
Use the UBP:

• $\mathcal{U}$ is an unconditional basis if and only if the span of $\mathcal{U}$ is dense and if the projections to subspaces generated by finite subsets of $\mathcal{U}$ are uniformly bounded.

• For unconditional bases the multiplier problem is trivial:

\[ \mathcal{U} \text{ is an unconditional basis if and only if the multiplier transformation } \sum_n c_n(f) u_n \mapsto \sum_n m(n) c_n(f) u_n \text{ is a bounded operator for all } m \in \ell^{\infty}(\mathbb{N}). \]
We say that $\mathcal{U}$ is an unconditional basic sequence on $X$ if $\mathcal{U}$ is an unconditional basis on $\text{span}(\mathcal{U})^X$.

For the Haar system the following notion is also useful. 
**Def.** $\mathcal{H}_d$ is a local basis of $X$ if $f = \sum c_n(f)u_n$ converges for all compactly supported $f$.

The statements about projection operators remain true, but the operator norms depend on the choice of a compact set $K$ in which all considered $f$ are supported.

One can also define the notions of local unconditional basis, local basic sequence and local unconditional basic sequence.
The Haar basis in $L^p(\mathbb{R})$

**Schauder (28):** $\mathcal{H}$ (with the natural lexicographic order) is a basis of $L^p([0, 1))$ when $1 \leq p < \infty$.

$$f = E_0 f + \sum_{j=0}^{\infty} \sum_{\mu=0}^{2^j-1} 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu}$$

for $f \in L^p([0, 1))$, with convergence in $L^p$.

- One works with conditional expectional operators $E_N$ associated to dyadic intervals of length $2^{-N}$.
- $E_{N+1} - E_N$ is the orthogonal projection to the space generated by the Haar functions with Haar frequency $2^N$.
- Billard (1970's): $\mathcal{H}$ is a Schauder basis on the Hardy space $h^1(\mathbb{T})$. 
Marcinkiewicz (37): \( \mathcal{H} \) is an \textit{unconditional basis} of \( L^p(\mathbb{R}) \) when \( 1 < p < \infty \).

- For \( f \in L^p \),
  \[
  f = \sum_{j=-1}^{\infty} \sum_{\mu \in \mathbb{Z}} 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu}
  \]
  with \textit{unconditional} convergence in \( L^p \).

Based on prior work of Paley, on square functions.

- Pełczynski (61): \( L^1 \) cannot be imbedded in a Banach space with an \textit{unconditional} basis.
Function spaces on $\mathbb{R}^d$, I.

Sobolev spaces $W^m_p$, $1 \leq p \leq \infty$, $m \in \mathbb{N}$.

$$\|f\|_{W^m_p} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p$$

Bessel potential space $L^p_s$ aka Sobolev space.

$$\|f\|_{L^p_s} = \|(I - \Delta)^{s/2} f\|_p$$

where $\mathcal{F}[(I - \Delta)^{s/2} f](\xi) = (1 + |\xi|^2)^{s/2} \hat{f}(\xi)$.

Note that for $1 < p < \infty$ we have $W^p_s = L^p_s$ and the $L^p_s$ interpolate with the complex method.

Since Haar functions are not smooth we are interested in these spaces for small $s$. 

$L^p$ Hölder classes $\Lambda(p, s) \equiv B_{p,\infty}^s$ For $0 < s < 1$, $1 \leq p \leq \infty$,

$$\|f\|_{B_{p,\infty}^s} = \|f\|_p + \sup_{h \neq 0} \frac{\|f(\cdot + h) - f\|_p}{|h|^s}.$$ 

Sobolev-Slobodecki spaces $B_{p,p}^s$. For $0 < s < 1$ let

$$\|f\|_{B_{p,p}^s} = \|f\|_p + \left( \int \int \frac{|f(x) - f(y)|^p}{|x - y|^{d + sp}} dx dy \right)^{1/p}$$

$B_{p,p}^s$ is also referred to as "Sobolev space of fractional order $s". But $B_{p,p}^s \neq L_p^p$ for $p \neq 2$. 
Consider \( \{P_k\}_{k=0}^{\infty} \), an inhomogeneous dyadic frequency decomposition. Aka Littlewood-Paley decomposition.

Let \( \phi_0 \in C^\infty_c((\mathbb{R}^d)^*) \), \( \phi_0 = 1 \) near 0.

\[
\widehat{P_0} f(\xi) = \phi_0(\xi) \hat{f}(\xi),
\]
\[
\widehat{P_k} f(\xi) = (\phi_0(2^{-k}\xi) - \phi_0(2^{1-k}\xi)) \hat{f}(\xi), \quad k \geq 1.
\]

- Localization to frequencies of size \( \approx 2^k \).

Then, for \( 1 < p < \infty \)

\[
\| f \|_{L^p_s} = \left\| \left( \sum_{k=0}^{\infty} 2^{2ks} |P_k f|^2 \right)^{1/2} \right\|_p.
\]

by standard singular integral theory (in a Hilbert-space setting).
"Function spaces" as subspaces of tempered distributions via Fourier analytic definitions:

**Besov-Nikolskij-Taibleson spaces**, \(0 < p, q \leq \infty\).

\[
\|f\|_{B^s_{p,q}} = \left\| \{2^{ks} P_k f \} \right\|_{\ell^q(L^p)}
\]

**Triebel-Lizorkin spaces**. \(0 < p < \infty, 0 < q \leq \infty\).

\[
\|f\|_{F^s_{p,q}} = \left\| \{2^{ks} P_k f \} \right\|_{L^p(\ell^q)}
\]

Note \(F^s_{p,2} = L^p_s\), \(1 < p < \infty\). Hardy-Sobolev \(H^s_p\) when \(p > 0\).

There is an extension to \(p = \infty\), so that \(F^0_{\infty,2} = BMO\), using \(BMO\)-like norms in the general case (cf. the Chang-Wilson-Wolff theorem and Frazier-Jawerth definitions).
Motivated by the Hardy space results of Fefferman-Stein, Peetre (1975) introduced maximal functions on distributions with bounded Fourier theorems about maximal functions support. Assume $f$ Schwartz and $\hat{f}$ supported in a set of diameter 1. Then

$$\sup_{x \in \mathbb{R}^d} \frac{|f(x + h)|}{(1 + |h|)^{d/r}} \lesssim (M_{HL}[|f|])^{1/r}.$$  

Summary of proof: One proves first

$$\sup_{x \in \mathbb{R}^d} \frac{|
abla f(x + h)|}{(1 + |h|)^{d/r}} \lesssim \sup_{x \in \mathbb{R}^d} \frac{|f(x + h)|}{(1 + |h|)^{d/r}}$$  

and then relies on a mean value inequality

$$|g(x)| \leq c_1 \delta \sup_{B_\delta(x)} |\nabla g| + c_2 \left( \text{av}_{B_\delta(x)} |g|^r \right)^{1/r}.$$
• Assume that $\hat{f}_k \in S'$ is supported on a set of diameter $R_k$. Let

$$M_{k,A}f_k(x) = \sup_{h \in \mathbb{R}^d} \frac{|f_k(x + h)|}{(1 + R_k|h|)^A}.$$ 

Then

$$\|\{M_{k,A}f_k\}\|_{\ell_q(L^p)} \lesssim_A \|\{f_k\}\|_{\ell_q(L^p)}, \quad A > d/p.$$ 

$$\|\{M_{k,A}f_k\}\|_{L^p(\ell_q)} \lesssim_A \|\{f_k\}\|_{L^p(\ell_q)}, \quad A > d/p, A > d/q.$$ 

One can use Fefferman-Stein vector-valued extension of the Hardy-Littlewood maximal theorem.

Is the additional condition on $q$ needed?
Assume that $\hat{f}_k \in S'$ is supported on a set of diameter $R_k$. Let

$$M_{k,A}f_k(x) = \sup_{h \in \mathbb{R}^d} \frac{|f_k(x + h)|}{(1 + R_k|h|)^A}.\]

Then

$$\|\{M_{k,A}f_k\}\|_{\ell_q(L^p)} \lesssim_A \|\{f_k\}\|_{\ell_q(L^p)}, \quad A > d/p.\]

$$\|\{M_{k,A}f_k\}\|_{L^p(\ell^q)} \lesssim_A \|\{f_k\}\|_{L^p(\ell^q)}, \quad A > d/p, A > d/q.\]

One can use Fefferman-Stein vector-valued extension of the Hardy-Littlewood maximal theorem.

Is the additional condition on $q$ needed? Yes, no matter what the $R_k$ are. (Christ, S., PLMS 2006).
Consider Triebel-Lizorkin spaces $F_{p,q}^s$, Besov spaces $B_{p,q}^s$.

**Q1:** For which spaces is $\mathcal{H}_d$ a basic sequence?

**Q2:** For which spaces is $\mathcal{H}_d$ a Schauder basis?

**Q3:** For which spaces is $\mathcal{H}_d$ an unconditional basis?

**Q4:** Haar system on unit cube or on $\mathbb{R}^d$: Does it matter for the outcomes?

- **Obvious necessary condition:** The Haar functions must belong to the space (mostly $s < 1/p$).
- **Other necessary conditions by duality** (e.g. mostly $s > -1 + 1/p$ when $1 < p < \infty$).
- **Interpolation** gives additional restrictions for cases with $p \leq 1$.
- **We often disregard** the cases $p = \infty$ or $q = \infty$ (Schwartz functions are not dense).
Some references to prior work

- Triebel (73), (78): $\mathcal{H}_d$ is an (unconditional) basis on $B_{p,q}^s$ if

$$\max\left\{ \frac{d}{p} - d, \frac{1}{p} - 1 \right\} < s < \min\left\{ \frac{1}{p}, 1 \right\}.$$  

Result is sharp up to endpoints. Secondary smoothness parameter $q$ plays no role.

- Many more results on splines, wavelets in Besov spaces (Ciesielski, Figiel, Ropela, Meyer, Sickel, Bourdaud, Oswald).

2010: Triebel’s monograph:

$\mathcal{H}_d$ is an unconditional basis on $F_{p,q}^s$ if

$$\max\left\{ \frac{1}{p} - 1, \frac{1}{q} - 1, \frac{d}{p} - d, \frac{d}{q} - d \right\} < s < \min\left\{ \frac{1}{p}, \frac{1}{q}, 1 \right\}.$$  

Q: Is the additional restriction on $q$ necessary?
Figure: Parameter domains for the Haar system in $F_{p,q}^s$ spaces on $\mathbb{R}^d$, $1 < q < \infty$, here $q = 2$.

$\mathcal{H}_d$ is unconditional basis for $B_{p,q}^s$: interior of the entire domain.
More about this on Thursday.