Basis properties of the Haar system in various function spaces, II.

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- Based on joint work with Gustavo Garrigós and Tino Ullrich

Questions for function spaces measuring smoothness

$\mathcal{H}_d$: System of Haar functions in $\mathbb{R}^d$ (or $\mathbb{T}^d$).
Consider Triebel-Lizorkin spaces $F^s_{p,q}$, Besov spaces $B^s_{p,q}$.

Q1: For which spaces is $\mathcal{H}_d$ a basic sequence?

Q2: For which spaces is $\mathcal{H}_d$ a Schauder basis?

Q3: For which spaces is $\mathcal{H}_d$ an unconditional basis?

Q4: Haar system on unit cube or on $\mathbb{R}^d$: Does it matter for the outcomes?

- Q3 refers to $\mathcal{H}^d$ as a set of functions.

- Q1, Q2, Q4 refer to specific enumerations of $\mathcal{H}^d$.

Pre-2014 results

• Triebel (73), (78): $\mathcal{H}_d$ is an (unconditional) basis on $B^s_{p,q}$ if

$$\max\{\frac{d}{p} - d,\ \frac{1}{p} - 1\} < s < \min\{\frac{1}{p}, 1\}.$$

Secondary smoothness parameter q plays no role. Result is sharp up to endpoints.

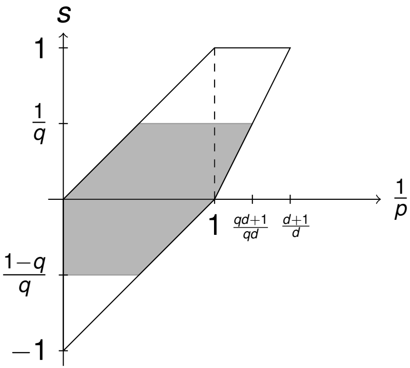
• Many more results on splines, wavelets in Besov spaces (Ciesielski, Figiel, Ropela, Meyer, Sickel, Bourdaud, Oswald).

2010: Triebel's monograph :
$\mathcal{H}_d$ is an unconditional basis on $F^s_{p,q}$ if

$$\max\{\frac{1}{p} - 1,\ \frac{1}{q} - 1,\ \frac{d}{p} - d, \frac{d}{q} - d\} < s < \min\{\frac{1}{p},\ \frac{1}{q},\ 1\}.$$

Q: Is the additional restriction on q necessary?

A recurring picture



• $\mathcal{H}_d$ is unconditional basis for $B^s_{p,q}$: interior of the entire domain.

• Tomorrow's topic: On $F^s_{p,q}$ unconditional basis property only holds in the grey region.

Schauder basis (interior of the figure)

Theorem (GSU)

Let $0 < q < \infty$, and $p > \frac{d}{d+1}$. Assume that

$$-1 + \frac{1}{p} < s < \frac{1}{p} \quad \text{if } p > 1,$$
$$-d + \frac{d}{p} < s < 1 \quad \text{if } \frac{d}{d+1} < p \leq 1.$$

Then $\mathcal{U}$ is a Schauder basis of $F^s_{p,q}$.

- This result refers to admissible enumerations $\mathcal{U}$ of $\mathcal{H}_d$.
- Analogous result is true on $\mathbb{T}$ or $\mathbb{T}^d$ where the lexicographic order can be used as the standard enumeration.
- Recall: For Besov spaces, Triebel had proved the result with unconditionality, so then admissibility is irrelevant.

Admissible enumerations

Admissibility means roughly: Mimicking the lexicographic order for Haar functions on the unit interval.

In endpoint cases one has to be careful with the definition of admissibility (characteristic function of cubes may not be pointwise multipliers). Here we use:

Def. An enumeration $\mathcal{U} = \{u_1, u_2, ...\}$ of the Haar system is (strongly) *admissible* if the following condition holds for some $b \in \mathbb{N}$. Whenever

u_n, $u_{n'}$ supported in 5-fold dilate of a dyadic unit cube,

$$|\text{supp }(u_n)| \geq 2^b |\text{supp } u_{n'}|$$

then $n \leq n'$.

Conditional expectations

Key is to understand boundedness properties for the conditional expectional operators $\mathbb{E}_N$ associated to dyadic intervals of length 2^{-N}.

Recall: $\mathbb{E}_{N+1} - \mathbb{E}_N$ is the orthogonal projection to the space generated by the Haar functions with Haar frequency 2^N.

Then a main ingredient for the Schauder basis property is

Theorem. Let $\max\{-d + d/p, -1 + 1/p\} < s < \min\{1/p, 1\}$, then

$$\|\mathbb{E}_N f\|_{F^s_{p,q}} \lesssim \|f\|_{F^s_{p,q}} .$$

Basic Idea: Approximate $\mathbb{E}_N$ (rough non-convolution approximation of the identity) with a nice convolution approximation of the identity $\Pi_N = \Phi_0(2^{-N}D)$, with suitable smooth compactly supported Φ_0 with $\int \Phi_0 dx = 1$.

Clearly for all s, p, q

$$\|\Pi_N f\|_{F^s_{p,q}} \lesssim \|f\|_{F^s_{p,q}} .$$

$\mathbb{E}_N - \mathit{\Pi}_N$ is better

Use embeddings to reduce to
Theorem. For p, s in the interior of the figure, any $r > 0$

$$\|\mathbb{E}_N f - \mathit{\Pi}_N f\|_{B^s_{p,r}} \lesssim \|f\|_{B^s_{p,\infty}} .$$

Note the continuous embeddings for $r \leq \min\{1, p, q\}$:

$$B^s_{p,r} \subset F^s_{p,r} \subset F^s_{p,q} \subset F^s_{p,\infty} \subset B^s_{p,\infty}.$$

- Approximation of $\mathbb{E}_N$ by $\mathit{\Pi}_N$ is harder in the endpoint cases.

Another kind of Littlewood-Paley decomposition

A standard tool is to decompose decompose

$$g = \sum_{j\geq 0} L_j P_j g = \sum_{j\geq 0} P_j L_j g.$$

L_j "Littlewood-Paley cutoff operators" which have appropriate compact support and, when $j > 0$, cancellation. For $j > 0$ we have

$$L_j g = 2^{jd}\psi(2^j\cdot) * g, \qquad \int \psi dx = 0 \text{ (or higher canc.)}$$

$$P_j g = 2^{jd}\beta(2^j\cdot) * g, \qquad \widehat{P_j g} \text{ supported where } |\xi| \approx 2^j.$$

• This is used in the Calderón reproducing formula, in the discrete form by Frazier and Jawerth.
• Use this for g being f or $\mathbb{E}_N f$.
• We may assume $\Pi_N = \sum_{j\leq N} L_j P_j$.

Decompose

O.B.d.A.(W.l.o.g.) $r \le \min\{1, p, q\}$:

$$\begin{aligned}
\left\|\mathbb{E}_N f - \Pi_N f\right\|_{B^s_{p,r}} \lesssim \; & \left\|\Big\{2^{ks} \sum_{j=N+1}^{\infty} P_k L_k \mathbb{E}_N L_j P_j f\Big\}\right\|_{\ell^r(L^p)} \\
& + \left\|\Big\{2^{ks} \sum_{j=0}^{N} P_k L_k (\mathbb{E}_N - I) L_j P_j f\Big\}\right\|_{\ell^r(L^p)} \\
\lesssim & \Big(\sum_{j=N+1}^{\infty} \sum_{k=0}^{\infty} 2^{ksr} \|P_k L_k \mathbb{E}_N L_j P_j f\|_p^r \Big)^{1/r} \\
& + \Big(\sum_{j=0}^{N} \sum_{k=0}^{\infty} 2^{ksr} \|P_k L_k (\mathbb{E}_N - I) L_j P_j f\|_p^r \Big)^{1/r}
\end{aligned}$$

- We need to show that this is $\lesssim \sup_j 2^{js} \|P_j f\|_p$.

$p \geq 1$

Lemma.

$$2^{ks}\|P_k L_k \mathbb{E}_N L_j P_j f\|_p \lesssim U_{p,s}(j,k,N) 2^{js}\|P_j f\|_p, \quad j \geq N$$
$$2^{ks}\|\|P_k L_k(\mathbb{E}_N - I) L_j P_j f\|_p \lesssim U_{p,s}(j,k,N) 2^{js}\|P_j f\|_p, \quad j \leq N$$

where

$$U_{p,s}(j,k,N) = \begin{cases} 2^{j(\frac{1}{p}-1-s)} 2^{k(s-\frac{1}{p})} 2^{N}, & j,k \geq N \\ 2^{j(1-s)} 2^{k(s-\frac{1}{p})} 2^{N(\frac{1}{p}-1)}, & j \leq N \leq k \\ 2^{j(1-s)} 2^{k(1+s)} 2^{-2N}, & j,k \leq N \\ 2^{j(\frac{1}{p}-1-s)} 2^{k(1+s)} 2^{-\frac{N}{p}}, & k \leq N \leq j. \end{cases}$$

• Good for estimates in the interior of the figure. Also some results at the boundary.

$p \leq 1$

Lemma.

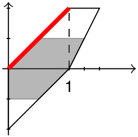
$$\begin{aligned} 2^{ks}\|P_k L_k \mathbb{E}_N L_j P_j f\|_p &\lesssim U_{p,s}(j,k,N) 2^{js}\|P_j f\|_p, \quad j \geq N \\ 2^{ks}\|\|P_k L_k(\mathbb{E}_N - I)L_j P_j f\|_p &\lesssim U_{p,s}(j,k,N) 2^{js}\|P_j f\|_p, \quad j \leq N \end{aligned}$$

where

$$U_{p,s}(j,k,N) = \begin{cases} 2^{j(\frac{d}{p}-d-s)} 2^{k(s-\frac{1}{p})} 2^{N(d-\frac{d-1}{p})} & \text{if } j,k > N \\ 2^{j(1-s)} 2^{k(s-\frac{1}{p})} 2^{N(\frac{1}{p}-1)} & \text{if } j \leq N < k \\ 2^{j(1-s)} 2^{k(s+d+1-\frac{d}{p})} 2^{N(\frac{d}{p}-d-2)} & \text{if } 0 \leq j,k \leq N \\ 2^{j(\frac{d}{p}-d-s)} 2^{k(s+d+1-\frac{d}{p})} 2^{-N} & \text{if } \ k \leq N < j. \end{cases}$$

• Good for estimates in the interior of the figure. Also some results at the boundary.

Endpoint case I: $1 < p < \infty$, $s = 1/p$.

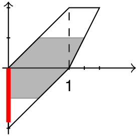


- Haar functions do not belong to $F^{1/p}_{p,q}$, $q \leq \infty$.
- Haar functions do not belong to $B^{1/p}_{p,q}$, $q < \infty$.
- But Haar functions belong to $B^{1/p}_{p,\infty}$.

Proposition

Let $1 < p < \infty$, $s = 1/p$. Then the operators $\mathbb{E}_N$ are uniformly bounded on $B^{1/p}_{p,\infty}$.

Endpoint case II: $p = \infty$, $-1 < s \leq 0$.

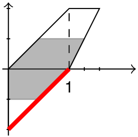


- Separability fails for $p = \infty$.

Proposition

Let $p = \infty$. (i) If $-1 < s < 0$ then the operators $\mathbb{E}_N$ are uniformly bounded on $B^s_{\infty,q}$, $0 < q < \infty$.
(ii) If $-1 < s \leq 0$ then the operators $\mathbb{E}_N$ are uniformly bounded on $B^s_{\infty,\infty}$.
(iii) If $-1 < s \leq 0$ then the operators $\mathbb{E}_N$ are uniformly bounded on $F^s_{\infty,q}$, $0 < q < \infty$.

Endpoint case III: $1 < p < \infty$, $s = 1/p - 1$.



Let $\mathcal{U}$ an admissible enumeration.

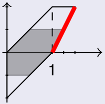
Theorem

Let $1 < p < \infty$, $s = 1/p - 1$.
(i) $\mathcal{U}$ not a basic sequence on $F^s_{p,q}$, $q > 0$, or $B^s_{p,q}$, for any $q > 1$.
(ii) Schauder basis property on $B^s_{p,q}(\mathbb{R}^d)$ fails for some admissible $\mathcal{U}$, any $q > 0$.
(iii) All admissible $\mathcal{U}$ are local Schauder bases on $B^s_{p,q}(\mathbb{R}^d)$ if and only if $0 < q \leq 1$.

- In (iii) unconditionality fails.
- In the cases $1 < p, q < \infty$ negative results follow by duality.

Endpoint case IV: $p \leq 1$, $s = d/p - d$

Theorem

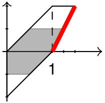


Let $\frac{d}{d+1} < p \leq 1$, $s = \frac{d}{p} - d$. *Then*

(i) All admissible $\mathcal{U}$ *are Schauder bases on* $F^s_{p,q}$ *for* $0 < q < \infty$, *and basic sequences on* $F^s_{p,\infty}$.
(ii) All admissible $\mathcal{U}$ *are Schauder bases (basic sequences) on* $B^s_{p,q}$, $s = \frac{d}{p} - d$, *if and only if* $q = p$.
(iii) All admissible $\mathcal{U}$ *are local Schauder bases (basic sequences) on* $B^s_{p,q}$, $s = \frac{d}{p} - d$, *if and only if* $0 < q \leq p$, *but uniform boundedness of the* $\mathbb{E}_N$ *fails for* $p < q \leq 1$.

- In all cases above $\mathcal{H}_d$ is not an unconditional basis.
- The positive result in (iii), on $[0, 1)^d$ was also obtained by Oswald.

Endpoint case IV, cont.



Def. Let Q be a large cube. X function space.

$$Op(T, X, Q) := \sup \left\{ \|Tf\|_X : \|f\|_X \leq 1, \mathrm{supp}\,(f) \subset Q \right\}.$$

Theorem

Let $\frac{d}{d+1} < p \leq 1$, $s = \frac{d}{p} - d$. Then for cubes of side length ≥ 1, and $p < q \leq 1$,

$$Op(\mathbb{E}_N, B^{\frac{d}{p}-d}_{p,q}, Q) \approx (2^{Nd}|Q|)^{1/p-1/q}.$$

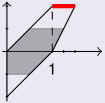
Ex.: Let $g_l(x) = 2^{ld}\eta(2^l x)$, $\int \eta = 0$, N large, $\{a_m\} \in \ell^q$.
Enumerate $\mathbb{Z}_d = \{\mathfrak{z}_m : m = 1, 2, \dots\}$. Set

$$F_N(x) := \sum a_m g_{N+m}(x - 2^{-N}\mathfrak{z}_m).$$

Endpoint case V: $s = 1$.

Let $\mathcal{U}$ be admissible enumeration of $\mathcal{H}_d$.

Theorem



Let $\frac{d}{d+1} \leq p < 1$.

(i) $\mathbb{E}_N$ *are uniformly bounded on* $F^1_{p,q}$ *if and only if* $0 < q \leq 2$.

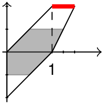
(ii) $\mathbb{E}_N$ *are uniformly bounded on* $B^1_{p,q}$ *if and only if* $0 < q \leq p$.

(iii) span $(\mathcal{H}_d)$ *is not dense on these spaces.*

(iv) All admissible $\mathcal{U}$ *are basic sequences in case (i) and local basic sequences in case (ii), global if* $p = q$.

• There are Schwartz functions for which $E_N f \not\to f$ in all spaces with $s = 1$ (also observed by Oswald).

Endpoint case V: $\mathbb{E}_N$ for $s = 1$, cont.

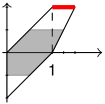


We do not want to separately estimate the contributions for $P_j f$ when $j \le N$.

Instead we use for $\frac{d}{d+2} < p < 1$, $r > 0$,

$$\Big(\sum_k \big[2^k \| L_k (I - \mathbb{E}_N) \Pi_N f \|_p \big]^r \Big)^{1/r} \lesssim \| \nabla f \|_{h^p} \approx \| f \|_{F^1_{p,2}}.$$

Endpoint case V: $\mathbb{E}_N$ for $s = 1$, cont.



Observe independence of Q, $|Q| > 1$, in:

Theorem

Let $\frac{d}{d+1} \le p < 1$ (or $p = 1, q = \infty$ in B-case). Then

$$\text{(i)} \quad Op(\mathbb{E}_N, B^1_{p,q}) \approx N^{1/p-1/q}, \quad p \le q \le \infty.$$

$$\text{(ii)} \quad Op(\mathbb{E}_N, F^1_{p,q}) \approx N^{1/2-1/q}, \quad 2 \le q \le \infty.$$

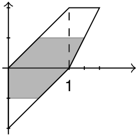
Example for (ii): $f_N(x) = \sum_{N/4<j<N/2} (\pm 1) 2^{-j} e^{2\pi i 2^j x} \psi(x)$ for random choices of sequences of ± 1.

Two unanswered questions:

- Is $\operatorname{span}(\mathcal{H}_d)$ dense in $B^1_{p,q}$ when $q > p$?
- Is $\operatorname{span}(\mathcal{H}_d)$ dense in $F^1_{p,q}$ when $q > 2$?

Failure of unconditionality in $F^s_{p,q}(\mathbb{R})$: A multiplier question for $p, q \geq 1$

On Friday we consider the question when $\mathcal{H}^d$ is an unconditional basis, with emphasis on counterexamples.



$$T_m f := \sum_{j=0}^{\infty} m(j) \sum_{\mu} 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu} = \sum_{j=0}^{\infty} m(j) \mathbb{D}_j f$$

where $\mathbb{D}_j = \mathbb{E}_{j+1} - \mathbb{E}_j$.
Recall: $\mathcal{H}_1$ unconditional basis $\iff$ every bounded sequence m is a multiplier.
Q: What are the conditions on m that T_m is bounded on $F^s_{p,q}$ for (p^{-1}, s) in the non-shaded regions?

Multiplier question, II

V^u: u-variation space:

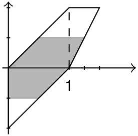
$$\|m\|_{V^u} = \|m\|_\infty + \sup_N \sup_{j_1<\cdots<j_N} \Big(\sum_{i=1}^{N-1} |m(j_{i+1}) - m(j_i)|^u \Big)^{1/u}$$

By a summation by parts argument it is easy to see: If the $\mathbb{E}_N$ are uniformly bounded on $\mathcal{X}$ then

$$\|T_m\|_{\mathcal{X}} \lesssim \|m\|_{V_1} \|f\|_{\mathcal{X}}.$$

Can one do better?

Multiplier question, III



Theorem

Let $1 < p < q < \infty$ *and* $1/q \le s < 1/p$*. Then*

$$\|T_m f\|_{F^s_{p,q}} \le C\|m\|_{V_u}\|f\|_{F^s_{p,q}}, \quad 1/u > s - 1/q\,.$$

Essentially sharp up to endpoints: Lower bounds for Haar projection numbers in [SU] give the existence of sets $E \subset 2\mathbb{N}$ depending on s such that $\#E \ge 2^N$, and thus $\|\mathbb{1}_E\|_{V^u} \ge 2^{N/u}$, and such that

$$\|T_{\mathbb{1}_E}\|_{F^s_{p,q}\to F^s_{p,q}} \gtrsim \begin{cases} 2^{N(s-\frac{1}{q})} & \text{if } \frac{1}{q} < s < \frac{1}{p}, \\ N & \text{if } \frac{1}{q} = s < \frac{1}{p}. \end{cases}$$

Multipliers IV: Variation norms and interpolation

We want to interpolate but variation norms cannot be efficiently interpolated (?).

• There is a related function space R^u such that

$$V^{\tilde{u}} \subset R^u \subset V^u, \qquad \tilde{u} < u.$$

Def. We say that g belongs to the class r^u if $g = \sum_\nu c_\nu \mathbb{1}_{I_\nu}$ where $(\sum_\nu |c_\nu|^u)^{1/u} \leq 1$.

Def. We say that h belongs to R^u if m can be written as

$$h = \sum_n a_n h_n$$

with $\sum |a_n| < \infty$ and the norm is given by $\inf \sum |a_n|$ where the inf is taken over all such representations.

• Since we don't prove an endpoint result we can reduce to an interpolation for ℓ^u spaces.

• This is sketched in a paper by Coifman, Rubio de Francia, Semmes (1988).