Basis properties of the Haar system in various function spaces, III.

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- Based on joint work with Gustavo Garrigós and Tino Ullrich
$\mathcal{H}$ is an unconditional basis on $F_{p,q}^s$ if

$$\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}.$$
Theorem (SU-MZ2017)

Let $1 < p, q < \infty$.

$\mathcal{H}$ is an unconditional basis on $F^s_{p,q}$ if and only if

$$\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}.$$ 

As a byproduct of the proof we also get

Theorem

For $1 < p, q < \infty$ we have $F^{s,\text{dyad}}_{p,q} = F^s_{p,q}$ if and only if

$$\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}.$$
X will be some Sobolev or Triebel-Lizorkin space.

For $E \subset \mathcal{H}_d$ let $HF(E) \subset \mathbb{N}$ be the Haar frequency set of $E$.

For any $A \subset \{2^n : n = 0, 1, \ldots \}$, set

$$G(X, A) := \sup \{ \| P_E \|_{X \to X} : E \subset \mathcal{H}, HF(E) \subset A \}.$$

Q1. How fast can $G(X, A)$ grow if $\# A$ grows?

Q2. How fast must $G(X, A)$ grow if $\# A$ grows?

Define, for $\lambda \in \mathbb{N}$, the upper and lower Haar projection numbers

$$\gamma^*(X; \lambda) := \sup \{ G(X, A) : \# A \leq \lambda \},$$

$$\gamma_*(X; \lambda) := \inf \{ G(X, A) : \# A \geq \lambda \}.$$
Behavior of $\gamma_*$ and $\gamma^*$

**Theorem**

Let $1 < p < q < \infty$, $1/q < s < 1/p$. Then

$$\gamma_*(F_p^s; \lambda) \approx \gamma^*(F_p^s; \lambda) \approx \lambda^{s-1/q}$$

In other words $\mathcal{G}(F_p^s, A) \approx (#A)^{s-1/q}$.

**Theorem (Endpoint)**

**Thm.** Let $1 < p < q < \infty$, $s = 1/q$. Then for large $\lambda$

$$\gamma^*(F_p^{1/q}; \lambda) \approx \log \lambda$$

$$\gamma_* (F_p^{1/q}; \lambda) \approx (\log \lambda)^{1/q'}$$

- Similar statements in the dual situation, i.e. $q < p$ and $-1/p' < s \leq -1/q'$.
- Proofs are done in the dual setting.
Idea for $G(F_{p,q}^s, A) \gtrsim (\#A)^{-s-1/q'}$ when $-1 < s < -1/q'$, $q < p$

Assume $d = 1$. Given a set $A \subset \{2^j\}_{j \in \mathbb{N}}$ of Haar frequencies, $N \gg 1$, and $\text{card}(A) \approx c2^N$.

- Let $E$ be the set of Haar functions supported in $[0, 1]$ with Haar frequencies in $A$. We shall see that we can split $E = E^{(1)} \cup E^{(2)}$ (disjoint union) so that

$$\| P_{E^{(1)}} - P_{E^{(2)}} \|_{F_{p,q}^s \rightarrow F_{p,q}^s} \gtrsim 2^N(-s-1/q').$$

The splitting will be random. Note that the operator norm of either $P_{E^{(1)}}$ or $P_{E^{(2)}}$ is $\gtrsim 2^N(-s-1/q').$

We need to construct $f$ with $\| f \|_{F_{p,q}^s} \leq 1$ and

$$\| P_{E^{(1)}} f - P_{E^{(2)}} f \|_{F_{p,q}^s} \gtrsim 2^N(-s-1/q').$$
Let
\[\mathcal{S} = \{(l, \nu) : 2^{l-N} \in A', \, \nu \in 2^N \Z, \, 2^{-l}\nu \in [0, 1]\}\]
and let \(\mathcal{S}_l\) be the slice for fixed \(l\).

Let
\[f = \sum_{2^{l-N} \in A'} f_l =: \sum_{(l, \nu) \in \mathcal{S}} (\pm 1)2^{-ls} \eta_{l, \nu}\]

where \(\eta_{l, \nu}\) are suitable "bump" functions of width \(2^{-l}\), located near \(2^{-l}\nu\), with sufficiently many vanishing moments.

Note: For fixed \(l\), "bumps" are \(2^N - l\) separated.

- Assume \(q < p < \infty\), \(-1 < s < -1/q'\). Then one has (cf. [Christ-S., PLMS 06]) (uniformly in choices of signs)
\[\|f\|_{F^s_{p, q}} \lesssim 1.\]

This is easy for \(p = q\) but requires a proof for \(q > p\). Later.
If $p > q$ and $f$ supported in $[0, 1]$

$$\| P_{E(1)} f - P_{E(2)} f \|_{F_{p,q}^s} \gtrsim \| P_{E(1)} f - P_{E(2)} f \|_{F_{q,q}^s}$$

and thus we estimate the $F_{q,q}^s$ norm from below.

Let $(j, l, \nu, \mu) \rightarrow n(j, l, \nu, \mu)$ be bijective and consider the Rademacher functions $r_n$. We need to show that for one $t$ (i.e. one choice of signs in $j, l, \nu, \mu$)

$$\left( \sum_k 2^{ksq} \left\| \sum_{(l,\nu)\in \mathcal{S}} \sum_{j \in A'} 2^j \sum_{\mu=0}^{2^j-1} r_{n(j,l,\nu,\mu)}(t) 2^{-ls} \langle \eta_l, \nu, h_j, \mu \rangle h_j, \mu^* \psi_k \right\|_q \| \right)^{1/q}$$

$$\gtrsim 2^{N(-s-1/q')}.$$ By averaging and Khinchine’s inequality it suffices to show

$$\left( \sum_k 2^{ksq} \left( \sum_{(l,\nu)\in \mathcal{S}} \sum_{j \in A'} \sum_{\mu=0}^{2^j-1} \left| 2^j 2^{-ls} \langle \eta_l, \nu, h_j, \mu \rangle h_j, \mu^* \psi_k \right|^2 \right)^{1/2} \|_q \right)^{1/q}$$

$$\gtrsim 2^{N(-s-1/q')}.$$ Keep only those terms $j = k, l = k + N.$
Keeping only terms with $j = k$, $l = k + N$ and using quasi-disjointness in $(\mu, \nu)$: It suffices to show and one easily gets

$$\left(\sum_{k \in A} 2^{ks} \sum_{\mu=0}^{2^k-1} \left\|2^k 2^{-(k+N)s} \langle \eta_{k+N, \nu(\mu)}, h_{k,\mu} \rangle h_{k,\mu} * \psi_k \right\|_q\right)^{1/q} \gtrsim 2^{N(-s-1/q')}.$$ 

There are also deterministic example where one has to be much more careful in the estimation for the lower bound ([SU]-constr.appr).

Concretely: show a lower bound for terms $j = k$, $l = k + N$ and a (smaller!) upper bound for all other terms. Possible with additional separation assumptions on subsets of $A$. 

Using the moment and support conditions for the $\eta_{l,\nu}$, and standard maximal estimates, one reduces to

$$\left\| \left( \sum_{l-N \in A} \left[ \sum_{\nu \in S_l} \mathbb{1}_{l,\nu} \right]^{q} \right)^{1/q} \right\|_p \leq C(p, q)$$

The $l_{l,\nu}$ are $2^{-l}$-intervals separated by $2^{N-l}$

It suffices to check this for $p = mq$, $m = 1, 2, 3, \ldots$. Immediate when $m = 1$.

Now w.l.o.g $q = 1$ and one checks

$$\int \left[ \sum_{l} \sum_{\nu \in S_l} \mathbb{1}_{l,\nu} \right]^{m} dx \leq B(m)$$

- The functions $\mathbb{1}_{l,\nu}$ are not independent, but have low correlation.
Alternatively (see [SU-MZ]):

When \( m \to \infty \) then \( B(m) \to \infty \) and so there will be no \( L^\infty \to L^\infty \) bound. But one can show

\[
\left\| \sum_I \sum_{\nu \in C_I} \mathbb{1}_{I_i, \nu} \right\|_{BMO} \lesssim C
\]

and use that \( L^1 \) and \( BMO \) can be interpolated via the complex method to yield \( L^q \).
Answer: It depends on the density of \( \log_2(A) = \{ k : 2^k \in A \} \) on intervals of length \( \sim \log_2(\#A) \). Here \( \#A \geq 2 \). Define

\[
\overline{Z}(A) = \max_{n \in \mathbb{Z}} \# \{ k : 2^k \in A, |k - n| \leq \log_2 \#A \},
\]

\[
\underline{Z}(A) = \min_{2^n \in A} \# \{ k : 2^k \in A, |k - n| \leq \log_2 \#A \}.
\]

Remarks: (i) \( 1 \leq \underline{Z}(A) \leq \overline{Z}(A) \leq 1 + 2 \log_2 \#A \).
(ii) \( \overline{Z}(A) = O(1) \) when \( \#A \approx 2^N \) and \( \log_2(A) \) is \( N \)-separated.
(iii) For \( A = [1, 2^N] \cap \mathbb{N} \), we have \( \overline{Z}(A) \geq N \).

Theorem

For \( 1 < p < q < \infty \),

\[
\overline{Z}(A)^{1 - \frac{1}{q}} \lesssim \frac{G(F_{p,q}^{1/q}, A)}{1} \lesssim \overline{Z}(A)^{1 - \frac{1}{q}}.
\]
On Friday we consider the question when $\mathcal{H}^d$ is an unconditional basis, with emphasis on counterexamples.

\[
T_m f := \sum_{j=0}^{\infty} m(j) \sum_{\mu} 2^j \langle f, h_j, \mu \rangle h_j, \mu = \sum_{j=0}^{\infty} m(j) D_j f
\]

where $D_j = E_{j+1} - E_j$.

Recall: $\mathcal{H}_1$ unconditional basis $\iff$ every bounded sequence $m$ is a multiplier.

Q: What are the conditions on $m$ that $T_m$ is bounded on $F^s_{p,q}$ for $(p^{-1}, s)$ in the non-shaded regions?
$V^u$: $u$-variation space:

$$\|m\|_{V^u} = \|m\|_\infty + \sup_N \sup_{j_1 < \cdots < j_N} \left( \sum_{i=1}^{N-1} |m(j_{i+1}) - m(j_i)|^u \right)^{1/u}$$

By a summation by parts argument it is easy to see: If the $E_N$ are uniformly bounded on $X$ then

$$\|Tm\|_X \lesssim \|m\|_{V^1} \|f\|_X.$$ 

Can one do better?
Multiplier question, III

\[ \text{Theorem} \]

Let \( 1 < p < q < \infty \) and \( 1/q \leq s < 1/p \). Then

\[
\| T_m f \|_{F_p^s, q} \leq C \| m \|_{V_u} \| f \|_{F_p^s, q}, \quad 1/u > s - 1/q.
\]

Essentially sharp up to endpoints: Lower bounds for Haar projection numbers in [SU] give the existence of sets \( E \subset 2^\mathbb{N} \) depending on \( s \) such that \( \# E \geq 2^N \), and thus \( \| 1_E \|_{V_u} \geq 2^{N/u} \), and such that

\[
\| T_{1_E} \|_{F_p^s, q \rightarrow F_p^s} \gtrsim \begin{cases} 2^{N(s - 1/q)} & \text{if } \frac{1}{q} < s < \frac{1}{p}, \\ N & \text{if } \frac{1}{q} = s < \frac{1}{p}. \end{cases}
\]
We want to interpolate but variation norms cannot be efficiently interpolated (?).

- There is a related function space $R^u$ such that
  \[ V^{\tilde{u}} \subset R^u \subset V^u, \quad \tilde{u} < u. \]

**Def.** We say that $g$ belongs to the class $r^u$ if $g = \sum_{\nu} c_{\nu} 1_{I_{\nu}}$ where \((\sum_{\nu} |c_{\nu}|^u)^{1/u} \leq 1\).

**Def.** We say that $h$ belongs to $R^u$ if $m$ can be written as
  \[ h = \sum_n a_n h_n \]
with $\sum |a_n| < \infty$ and the norm is given by $\inf \sum |a_n|$ where the inf is taken over all such representations.

- Since we don’t prove an endpoint result we can reduce to an interpolation for $\ell^u$ spaces.
- This is sketched in a paper by Coifman, Rubio de Francia, Semmes (1988).