Basis properties of the Haar system in various function spaces, III.

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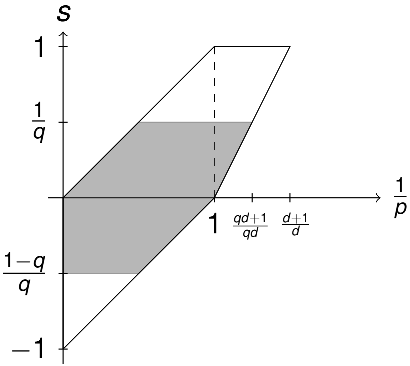
- Based on joint work with Gustavo Garrigós and Tino Ullrich

Triebel (2010) .

$\mathcal{H}$ is an unconditional basis on $F^s_{p,q}$ if

$$\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}.$$



Restrictions for unconditional basis property

Theorem (SU-MZ2017)

Let $1 < p, q < \infty$.
$\mathcal{H}$ *is an unconditional basis on* $F^s_{p,q}$ *if and only if*

$$\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}.$$

As a byproduct of the proof we also get

Theorem

For $1 < p, q < \infty$ *we have* $F^{s,\mathrm{dyad}}_{p,q} = F^s_{p,q}$ *if and only if* $\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}$.

Failure of unconditionality: Quantitative versions

X will be some Sobolev or Triebel-Lizorkin space.

For $E \subset \mathcal{H}_d$ let $HF(E) \subset \mathbb{N}$ be the Haar frequency set of E.

For any $A \subset \{2^n : n = 0, 1, \dots\}$, set

$$\mathcal{G}(X, A) := \sup\left\{\|P_E\|_{X \to X} : E \subset \mathcal{H},\, \mathrm{HF}(E) \subset A\right\}.$$

Q1. How fast can $\mathcal{G}(X, A)$ grow if $\#A$ grows?

Q2. How fast must $\mathcal{G}(X, A)$ grow if $\#A$ grows?

Define, for $\lambda \in \mathbb{N}$, the upper and lower *Haar projection numbers*

$$\gamma^*(X; \lambda) := \sup\left\{\mathcal{G}(X, A) : \#A \le \lambda\right\},$$
$$\gamma_*(X; \lambda) := \inf\left\{\mathcal{G}(X, A) : \#A \ge \lambda\right\}.$$

Behavior of γ_* and γ^*

Theorem

Let $1 < p < q < \infty$, $1/q < s < 1/p$. *Then*

$$\gamma_*(F^s_{p,q};\lambda) \approx \gamma^*(F^s_{p,q};\lambda) \approx \lambda^{s-1/q}$$

In other words $\mathcal{G}(F^s_{p,q}, A) \approx (\#A)^{s-1/q}$.

Theorem (Endpoint)

Thm. *Let* $1 < p < q < \infty$, $s = 1/q$. *Then for large* λ

$$\gamma^*(F^{1/q}_{p,q};\lambda) \approx \log\lambda$$
$$\gamma_*(F^{1/q}_{p,q};\lambda) \approx (\log\lambda)^{1/q'}$$

- Similar statements in the dual situation, i.e. $q < p$ and $-1/p' < s \leq -1/q'$.
- Proofs are done in the dual setting.

Idea for $\mathcal{G}(F^s_{p,q}, A) \gtrsim (\#A)^{-s-1/q'}$ when $-1 < s < -1/q'$, $q < p$

Assume $d = 1$. Given a set $A \subset \{2^j\}_{j\in\mathbb{N}}$ of Haar frequencies, $N \gg 1$, and $\mathrm{card}(A) \approx c2^N$.

- Let E be the set of Haar functions supported in $[0,1]$ with Haar frequencies in A. We shall see that we can split $E = E^{(1)} \cup E^{(2)}$ (disjoint union) so that

$$\left\| P_{E^{(1)}} - P_{E^{(2)}} \right\|_{F^s_{p,q} \to F^s_{p,q}} \gtrsim 2^{N(-s-1/q')}.$$

The splitting will be random. Note that the operator norm of either $P_{E^{(1)}}$ or $P_{E^{(2)}}$ is $\gtrsim 2^{N(-s-1/q')}$..
We need to construct f with $\|f\|_{F^s_{p,q}} \leq 1$ and

$$\|P_{E^{(1)}} f - P_{E^{(2)}} f\|_{F^s_{p,q}} \gtrsim 2^{N(-s-1/q')}.$$

The test functions f

Let

$$\mathfrak{S} = \{(l,\nu) : 2^{l-N} \in A',\, \nu \in 2^N\mathbb{Z},\, 2^{-l}\nu \in [0,1]\}$$

and let $\mathfrak{S}_l$ be the slice for fixed l.

Let

$$f = \sum_{2^{l-N}\in A'} f_l =: \sum_{(l,\nu)\in\mathfrak{S}} (\pm 1)2^{-ls}\eta_{l,\nu}$$

where $\eta_{\ell,\nu}$ are suitable "bump" functions of width 2^{-l}, located near $2^{-l}\nu$, with sufficiently many vanishing moments.

Note: For fixed l, "bumps" are 2^{N-l} separated.

• Assume $q < p < \infty$, $-1 < s < -1/q'$. Then one has (cf. [Christ-S., PLMS 06]) (uniformly in choices of signs)

$$\|f\|_{F^s_{p,q}} \lesssim 1.$$

This is easy for $p = q$ but requires a proof for $q > p$. Later.

Lower bound for $P_{E^{(1)}} - P_{E^{(2)}}$

If $p > q$ and f supported in $[0, 1]$

$$\|P_{E^{(1)}} f - P_{E^{(2)}} f\|_{F^s_{p,q}} \gtrsim \|P_{E^{(1)}} f - P_{E^{(2)}} f\|_{F^s_{q,q}}$$

and thus we estimate the $F^s_{q,q}$ norm from below.
Let $(j, l, \nu, \mu) \to n(j, l, \nu, \mu)$ be bijective and consider the Rademacher functions r_n. We need to show that for one t (i.e. one choice of signs in j, l, ν, μ)

$$\Big(\sum_k 2^{ksq} \Big\| \sum_{(l,\nu)\in\mathfrak{S}} \sum_{j\in A'} 2^j \sum_{\mu=0}^{2^j-1} r_{n(j,l,\nu,\mu)}(t) 2^{-ls} \langle \eta_{l,\nu}, h_{j,\mu}\rangle h_{j,\mu} * \psi_k \Big\|_q^q\Big)^{1/q}$$

$\gtrsim 2^{N(-s-1/q')}$. By averaging and Khinchine's inequality it suffices to show

$$\Big(\sum_k 2^{ksq} \Big\| \Big(\sum_{(l,\nu)\in\mathfrak{S}} \sum_{j\in A'} \sum_{\mu=0}^{2^j-1} |2^j 2^{-ls} \langle \eta_{l,\nu}, h_{j,\mu}\rangle h_{j,\mu} * \psi_k|^2\Big)^{1/2} \Big\|_q^q\Big)^{1/q}$$

$\gtrsim 2^{N(-s-1/q')}$. Keep only those terms $j = k$, $l = k + N$.

Keeping only terms with $j = k$, $l = k + N$ and using quasi-disjointness in (μ, ν): It suffices to show and one easily gets

$$\Big(\sum_{k \in A} 2^{ksq} \sum_{\mu=0}^{2^k-1} \Big\| 2^k 2^{-(k+N)s} |\langle \eta_{k+N,\nu(\mu)}, h_{k,\mu} \rangle h_{k,\mu} * \psi_k \Big\|_q^q \Big)^{1/q}$$

$$\gtrsim 2^{N(-s-1/q')}.$$

There are also deterministic example where one has to be much more careful in the estimation for the lower bound ([SU]-constr.appr).
Concretely: show a lower bound for terms $j = k$, $l = k + N$ and a (smaller!) upper bound for all other terms. Possible with additional separation assumptions on subsets of A.

Lower bounds for $f = \sum_l f_l$

Using the moment and support conditions for the $\eta_{l,\nu}$, and standard maximal estimates, one reduces to

$$\Big\| \Big(\sum_{l-N \in A} \Big[\sum_{\nu \in \mathfrak{S}_l} \mathbb{1}_{I_{l,\nu}} \Big]^q \Big)^{1/q} \Big\|_p \le C(p,q)$$

The $I_{l,\nu}$ are 2^{-l}-intervals separated by 2^{N-l}
It suffices to check this for $p = mq$, $m = 1, 2, 3, \dots$. Immediate when $m = 1$.

Now w.l.o.g $q = 1$ and one checks

$$\int \Big[\sum_l \sum_{\nu \in \mathfrak{S}_l} \mathbb{1}_{I_{l,\nu}} \Big]^m dx \le B(m)$$

• The functions $\mathbb{1}_{I_{l,\nu}}$ are not independent, but have low correlation.

BMO bound

Alternatively (see [SU-MZ]):
When $m \to \infty$ then $B(m) \to \infty$ and so there will be no $L^\infty \to L^\infty$ bound. But one can show

$$\| \sum_{l} \sum_{\nu \in \mathfrak{S}_l} \mathbb{1}_{I_{l,\nu}} \|_{BMO} \lesssim C$$

and use that L^1 and *BMO* can be interpolated via the complex method to yield L^q.

Endpoint: How does $\mathcal{G}(F^{1/q}_{p,q}, A)$ depend on A?

Answer: It depends on the density of $\log_2(A) = \{k : 2^k \in A\}$ on intervals of length $\sim \log_2(\#A)$. Here $\#A \geq 2$. Define

$$\overline{\mathcal{Z}}(A) = \max_{n\in\mathbb{Z}} \#\{k : 2^k \in A,\ |k-n| \leq \log_2 \#A\},$$

$$\underline{\mathcal{Z}}(A) = \min_{2^n\in A} \#\{k : 2^k \in A,\ |k-n| \leq \log_2 \#A\}.$$

Remarks: (i) $1 \leq \underline{\mathcal{Z}}(A) \leq \overline{\mathcal{Z}}(A) \leq 1 + 2\log_2 \#A$.
(ii) $\overline{\mathcal{Z}}(A) = O(1)$ when $\#A \approx 2^N$ and $\log_2(A)$ is N-separated.
(iii) For $A = [1, 2^N] \cap \mathbb{N}$ we have $\underline{\mathcal{Z}}(A) \geq N$.

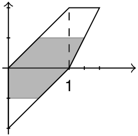
Theorem

For $1 < p < q < \infty$,

$$\underline{\mathcal{Z}}(A)^{1-\frac{1}{q}} \lesssim \frac{\mathcal{G}(F^{1/q}_{p,q}, A)}{(\log_2 \#A)^{\frac{1}{q}}} \lesssim \overline{\mathcal{Z}}(A)^{1-\frac{1}{q}}.$$

Failure of unconditionality in $F^s_{p,q}(\mathbb{R})$: A multiplier question for $p, q \geq 1$

On Friday we consider the question when $\mathcal{H}^d$ is an unconditional basis, with emphasis on counterexamples.



$$T_m f := \sum_{j=0}^{\infty} m(j) \sum_{\mu} 2^j \langle f, h_{j,\mu} \rangle h_{j,\mu} = \sum_{j=0}^{\infty} m(j) \mathbb{D}_j f$$

where $\mathbb{D}_j = \mathbb{E}_{j+1} - \mathbb{E}_j$.
Recall: $\mathcal{H}_1$ unconditional basis $\iff$ every bounded sequence m is a multiplier.
Q: What are the conditions on m that T_m is bounded on $F^s_{p,q}$ for (p^{-1}, s) in the non-shaded regions?

Multiplier question, II

V^u: u-variation space:

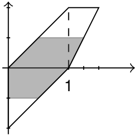
$$\|m\|_{V^u} = \|m\|_\infty + \sup_N \sup_{j_1 < \cdots < j_N} \Big(\sum_{i=1}^{N-1} |m(j_{i+1}) - m(j_i)|^u \Big)^{1/u}$$

By a summation by parts argument it is easy to see: If the $\mathbb{E}_N$ are uniformly bounded on $\mathcal{X}$ then

$$\|T_m\|_{\mathcal{X}} \lesssim \|m\|_{V_1} \|f\|_{\mathcal{X}}.$$

Can one do better?

Multiplier question, III



Theorem

Let $1 < p < q < \infty$ *and* $1/q \le s < 1/p$. *Then*

$$\|T_m f\|_{F^s_{p,q}} \le C\|m\|_{V_u}\|f\|_{F^s_{p,q}}, \quad 1/u > s - 1/q\,.$$

Essentially sharp up to endpoints: Lower bounds for Haar projection numbers in [SU] give the existence of sets $E \subset 2\mathbb{N}$ depending on s such that $\#E \ge 2^N$, and thus $\|\mathbb{1}_E\|_{V^u} \ge 2^{N/u}$, and such that

$$\|T_{\mathbb{1}_E}\|_{F^s_{p,q} \to F^s_{p,q}} \gtrsim \begin{cases} 2^{N(s-\frac{1}{q})} & \text{if } \frac{1}{q} < s < \frac{1}{p}, \\ N & \text{if } \frac{1}{q} = s < \frac{1}{p}. \end{cases}$$

Multipliers IV: Variation norms and interpolation

We want to interpolate but variation norms cannot be efficiently interpolated (?).

• There is a related function space R^u such that

$$V^{\tilde{u}} \subset R^u \subset V^u, \qquad \tilde{u} < u.$$

Def. We say that g belongs to the class r^u if $g = \sum_\nu c_\nu \mathbb{1}_{I_\nu}$ where $(\sum_\nu |c_\nu|^u)^{1/u} \leq 1$.

Def. We say that h belongs to R^u if m can be written as

$$h = \sum_n a_n h_n$$

with $\sum |a_n| < \infty$ and the norm is given by $\inf \sum |a_n|$ where the inf is taken over all such representations.

• Since we don't prove an endpoint result we can reduce to an interpolation for ℓ^u spaces.

• This is sketched in a paper by Coifman, Rubio de Francia, Semmes (1988).