Sampling discretization of integral norms. Lecture 1

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- Functions belong to a given function class.
  There are different settings and different ingredients, which play important role in this problem.

We begin with the Marcinkiewicz-type discretization.

Marcinkiewicz problem

Let Ω be a compact subset of $\mathbb{R}^d$ with the probability measure μ. We say that a linear subspace X_N of the $L_q(\Omega)$, $1 \le q < \infty$, admits the Marcinkiewicz-type discretization theorem with parameters m and q if there exist a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ and two positive constants $C_j(d, q)$, $j = 1, 2$, such that for any $f \in X_N$ we have

$$C_1(d,q)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \qquad (1)$$

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In the case $q = \infty$ we define L_∞ as the space of continuous on Ω functions and ask for

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We will also use a brief way to express the above property: the $\mathcal{M}(m,q)$ theorem holds for a subspace X_N or $X_N \in \mathcal{M}(m,q)$.

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$$P'(\mathbf{N}) := \Big\{ \mathbf{n} = (n_1, \dots, n_d), \qquad n_j - \quad \text{are natural numbers},$$

$$0 \le n_j \le 4N_j - 1, \qquad j = 1, \dots, d \Big\}$$

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$$0 \le n_j \le 4N_j - 1, \qquad j = 1, \dots, d\}$$

and set

$$\mathbf{x}(\mathbf{n}) := \left(\frac{\pi n_1}{2N_1}, \dots, \frac{\pi n_d}{2N_d}\right), \qquad \mathbf{n} \in P'(\mathbf{N}).$$

In the case $N_j = 0$ we assume $x_j(\mathbf{n}) = 0$. Denote $\overline{N} := \max(N, 1)$ and $v(\mathbf{N}) := \prod_{j=1}^d \overline{N}_j$. Then the following Marcinkiewicz-type discretization theorem is known for all $1 \le q \le \infty$: for any $f \in \mathcal{T}(\Pi(\mathbf{N}))$

$$C_1(d,q)\|t\|_q^q \le v(4\mathbf{N})^{-1} \sum_{\mathbf{n}\in P'(\mathbf{N})} \left|f(\mathbf{x}(\mathbf{n}))\right|^q \le C_2(d,q)\|t\|_q^q, \quad (5)$$

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which implies the following relation

$$\mathcal{T}(\Pi(\mathbf{N})) \in \mathcal{M}(v(4\mathbf{N}), q), \quad 1 \le q \le \infty.$$

Note that $v(4\mathbf{N}) \le C(d) \dim \mathcal{T}(\Pi(\mathbf{N}))$.

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$$C_1(d,q)\|t\|_q^q \le v(4\mathbf{N})^{-1} \sum_{\mathbf{n}\in P'(\mathbf{N})} \left|f(\mathbf{x}(\mathbf{n}))\right|^q \le C_2(d,q)\|t\|_q^q, \quad (5)$$

which implies the following relation

$$\mathcal{T}(\Pi(\mathbf{N})) \in \mathcal{M}(v(4\mathbf{N}), q), \quad 1 \le q \le \infty.$$

Note that $v(4\mathbf{N}) \le C(d) \dim \mathcal{T}(\Pi(\mathbf{N}))$. It is clear from the above construction that the set $\{\mathbf{x}(\mathbf{n}) : \mathbf{n} \in P'(\mathbf{N})\}$ depends substantially on $\mathbf{N}$. The main goal of this paper is to construct for a given q and M a set, which satisfies an analog of (5) for all $\mathbf{N}$ with $v(\mathbf{N}) \le M$.

Discretization for trigonometric polynomials in L_2

Let Q be a finite subset of $\mathbb{Z}^d$. We denote

$$\mathcal{T}(Q) := \{f : f = \sum_{\mathbf{k}\in Q} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}\}.$$

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The following result was obtained by VT, 2017.

Theorem (1; VT, 2017)

There are three positive absolute constants C_1, C_2, and C_3 with the following properties: For any $d \in \mathbb{N}$ and any $Q \subset \mathbb{Z}^d$ there exists a set of $m \le C_1|Q|$ points $\xi^j \in \mathbb{T}^d$, $j = 1, \dots, m$ such that for any $f \in \mathcal{T}(Q)$ we have

$$C_2\|f\|_2^2 \le \frac{1}{m}\sum_{j=1}^{m} |f(\xi^j)|^2 \le C_3\|f\|_2^2.$$

NOU Lemma

The above theorem is based on the following lemma from S. Nitzan, A. Olevskii, and A. Ulanovskii, 2016.

Lemma (NOU, 2016)

Let a system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_M$ *from* $\mathbb{C}^N$ *have the following properties: for all* $\mathbf{w} \in \mathbb{C}^N$ *we have* $\sum_{j=1}^M |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 = \|\mathbf{w}\|_2^2$ *and* $\|\mathbf{v}_j\|_2^2 = N/M, \quad j = 1, \dots, M$. *Then there is a subset* $J \subset \{1, 2, \dots, M\}$ *such that for all* $\mathbf{w} \in \mathbb{C}^N$

$$c_0 \|\mathbf{w}\|_2^2 \le \frac{M}{N} \sum_{j \in J} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \le C_0 \|\mathbf{w}\|_2^2,$$

where c_0 *and* C_0 *are some absolute positive constants.*

The above Lemma was derived from the following theorem from A. Marcus, D.A. Spielman, and N. Srivastava, 2015, which solved the **Kadison-Singer problem**.

Theorem (MSS, 2015)

Let a system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_M$ *from* $\mathbb{C}^N$ *have the following properties: for all* $\mathbf{w} \in \mathbb{C}^N$ *we have* $\sum_{j=1}^M |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 = \|\mathbf{w}\|_2^2$ *and* $\|\mathbf{v}_j\|_2^2 \le \epsilon$.
Then there exists a partition of $\{1, \dots, M\}$ *into two sets* S_1 *and* S_2*, such that for each* $i = 1, 2$ *we have for all* $\mathbf{w} \in \mathbb{C}^N$

$$\sum_{j \in S_i} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \le \frac{(1 + \sqrt{2\epsilon})^2}{2} \|\mathbf{w}\|_2^2.$$

Discretization for the trigonometric polynomials in L_1

Let $\Pi(\mathbf{N}) := [-N_1, N_1] \times \cdots \times [-N_d, N_d]$, $N_j \in \mathbb{N}$ or $N_j = 0$, $j = 1, \dots, d$, $\mathbf{N} = (N_1, \dots, N_d)$. The following result is obtained by VT, 2017.

Let $\Pi(\mathbf{N}) := [-N_1, N_1] \times \cdots \times [-N_d, N_d]$, $N_j \in \mathbb{N}$ or $N_j = 0$, $j = 1, \dots, d$, $\mathbf{N} = (N_1, \dots, N_d)$. The following result is obtained by VT, 2017.

Theorem (2; VT, 2017)

Let $d \in \mathbb{N}$. For any $n \in \mathbb{N}$ and any $Q \subset \Pi(\mathbf{N})$ with $\mathbf{N} = (2^n, \dots, 2^n)$ there exists a set of $m \le C_1(d)|Q|n^{7/2}$ points $\xi^j \in \mathbb{T}^d$, $j = 1, \dots, m$ such that for any $f \in \mathcal{T}(Q)$ we have

$$C_2(d)\|f\|_1 \le \frac{1}{m}\sum_{j=1}^{m}|f(\xi^j)| \le C_3(d)\|f\|_1.$$

Discretization of the uniform norm

We discussed in Kashin and Temlyakov, 2018, the following setting of the discretization problem of the uniform norm. Let $S_m := \{\xi^j\}_{j=1}^m \subset \mathbb{T}^d$ be a finite set of points. Clearly,

$$\|f\|_{L^\infty(S_m)} := \max_{1\le j\le m} |f(\xi^j)| \le \|f\|_\infty.$$

We are interested in estimating the following quantities

$$D(Q,m) := D(Q,m,d) := \inf_{S_m} \sup_{f\in\mathcal{T}(Q)} \frac{\|f\|_\infty}{\|f\|_{L^\infty(S_m)}},$$

$$D(N,m) := D(N,m,d) := \sup_{Q,|Q|=N} D(Q,m,d).$$

Simple remarks

Certainly, one should assume that $m \geq N$. Then the characteristic $D(Q, m)$ guarantees that there exists a set of m points S_m such that for any $f \in \mathcal{T}(Q)$ we have

$$\|f\|_\infty \leq D(Q, m)\|f\|_{L^\infty(S_m)}.$$

In the case $d = 1$ and $Q = [-n, n]$ classical Marcinkiewicz theorem gives for $m \geq 4n$ that $D([-n, n], 4n) \leq C$. Similar relation holds for $D([-n_1, n_1] \times \cdots \times [-n_d, n_d], (4n_1) \times \cdots \times (4n_d))$.

Lower bound

It was proved in Kashin and Temlyakov, 2018, that for a pair N, m, such that $m \asymp N$ we have $D(N,m) \asymp N^{1/2}$. We formulate this result as a theorem.

Theorem (KT, 2018)

For any constant $c \geq 1$ there exists a positive constant C such that for any pair of parameters N, m, with $m \leq cN$ we have

$$D(N,m) \geq CN^{1/2}.$$

Also, there are two positive absolute constants c_1 and C_1 with the following property: For any $d \in \mathbb{N}$ we have for $m \geq c_1 N$

$$D(N,m,d) \leq C_1 N^{1/2}.$$

Hyperbolic crosses

Recall that the set of hyperbolic cross polynomials is defined as

$$\mathcal{T}(N) := \mathcal{T}(N,d) := \left\{ f : \quad f = \sum_{\mathbf{k}\in\Gamma(N)} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})} \right\},$$

where $\Gamma(N)$ is the hyperbolic cross

$$\Gamma(N) := \Gamma(N,d) := \left\{ \mathbf{k} \in \mathbb{Z}^d : \quad \prod_{j=1}^{d} \max\{|k_j|, 1\} \le N \right\}.$$

Throughout this section, we define

$$\alpha_d := \sum_{j=1}^{d} \frac{1}{j} \qquad \text{and} \qquad \beta_d := d - \alpha_d.$$

We use the following notation here. For $\mathbf{x} \in \mathbb{T}^d$ and $j \in \{1, \dots, d\}$ we denote $\mathbf{x}^j := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$.

Main result

The following result was obtained by Dai, Prymak, Temlyakov, and Tikhonov, 2018.

Theorem (DPTT, 2018)

For each $d \in \mathbb{N}$ *and each* $N \in \mathbb{N}$ *there exists a set* $W(N,d)$ *of at most* $C_d N^{\alpha_d} (\log N)^{\beta_d}$ *points in* $[0,2\pi)^d$ *such that for all* $f \in \mathcal{T}(N)$*,*

$$\|f\|_\infty \le C(d) \max_{\mathbf{w} \in W(N,d)} |f(\mathbf{w})|.$$

Some historical remarks

It is well known that

$$\mathcal{T}(\Pi(N)) \in \mathcal{M}(C(d)N^d, \infty),$$

$$\Pi(N) := \{\mathbf{k} \in \mathbb{Z}^d : |k_j| \le N, j = 1, \dots, d\}.$$

In particular, this implies that

$$\mathcal{T}(N) \in \mathcal{M}(C(d)N^d, \infty).$$

Theorem DPTT shows that we can improve the above relation to

$$\mathcal{T}(N) \in \mathcal{M}(C(d)N^{\alpha_d}(\log N)^{\beta_d}, \infty).$$

Note that $\alpha_d \asymp \ln d$.

Lower bound

A trivial lower bound for m in the inclusion $\mathcal{T}(N) \in \mathcal{M}(m,\infty)$ is $m \ge \dim(\mathcal{T}(N)) \asymp N(\log N)^{d-1}$. The following nontrivial lower bound was obtained in Kashin and Temlyakov, 1998.

Theorem (KT, 1998)

Let a set $W \subset \mathbb{T}^2$ have a property:

$$\forall t \in \mathcal{T}(N) \qquad \|t\|_\infty \le b(\log N)^\alpha \max_{\mathbf{w}\in W} |t(\mathbf{w})|$$

with some $0 \le \alpha < 1/2$. Then

$$|W| \ge C_1 N \log N e^{C_2 b^{-2}(\log N)^{1-2\alpha}}.$$

In particular, Theorem KT with $\alpha = 0$ implies that a necessary condition on m for inclusion $\mathcal{T}(N) \in \mathcal{M}(m,\infty)$ is $m \ge \dim(\mathcal{T}(N))N^c$ with positive absolute constant c.

Upper bound

An operator T_N with the following properties was constructed in Temlyakov, 1993. The operator T_N has the form

$$T_N(f) = \sum_{j=1}^{m} f(\mathbf{x}^j)\psi_j(\mathbf{x}), \quad m \le c(d)N(\log N)^{d-1}, \quad \psi_j \in \mathcal{T}(N2^d)$$

and

$$T_N(f) = f, \quad f \in \mathcal{T}(N), \tag{3}$$

$$\|T_N\|_{L_\infty \to L_\infty} \asymp (\log N)^{d-1}. \tag{4}$$

Good points

Points $\{\mathbf{x}^j\}$ are from the Smolyak net. Properties (3) and (4) imply that all $f \in \mathcal{T}(N)$ satisfy the discretization inequality

$$\|f\|_\infty \leq C(d)(\log N)^{d-1} \max_{1\leq j\leq m} |f(\mathbf{x}^j)|.$$

Some remarks for the case $q = 2$

We describe the properties of the subspace X_N in terms of a system $\mathcal{U}_N := \{u_i\}_{i=1}^N$ of functions such that $X_N = \operatorname{span}\{u_i, i = 1, \dots, N\}$. In the case $X_N \subset L_2$ we assume that the system is orthonormal on Ω with respect to measure μ. In the case of real functions we associate with $x \in \Omega$ the matrix $G(x) := [u_i(x)u_j(x)]_{i,j=1}^N$. Clearly, $G(x)$ is a symmetric positive semi-definite matrix of rank 1. It is easy to see that for a set of points $\xi^k \in \Omega$, $k = 1, \dots, m$, and $f = \sum_{i=1}^N b_i u_i$ we have

$$\sum_{k=1}^m \lambda_k f(\xi^k)^2 - \int_\Omega f(x)^2 d\mu = \mathbf{b}^T \left(\sum_{k=1}^m \lambda_k G(\xi^k) - I \right) \mathbf{b},$$

where $\mathbf{b} = (b_1, \dots, b_N)^T$ is the column vector and I is the identity matrix.

Remarks continue

Therefore, the $\mathcal{M}^w(m,2)$ problem is closely connected with a problem of approximation (representation) of the identity matrix I by an m-term approximant with respect to the system $\{G(x)\}_{x\in\Omega}$. It is easy to understand that under our assumptions on the system $\mathcal{U}_N$ there exist a set of knots $\{\xi^k\}_{k=1}^m$ and a set of weights $\{\lambda_k\}_{k=1}^m$, with $m \le N^2$ such that

$$I = \sum_{k=1}^{m} \lambda_k G(\xi^k)$$

and, therefore, we have for any $X_N \subset L_2$ that

$$X_N \in \mathcal{M}^w(N^2, 2, 0).$$

Condition E

We begin with formulation of the Rudelson result from 1999. Let $\Omega_M = \{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu(x^j) = 1/M$, $j = 1, \dots, M$. Assume that $\{u_i(x)\}_{i=1}^N$ is a real orthonormal on Ω_M system satisfying the following condition:
Condition E. For all j

$$\sum_{i=1}^N u_i(x^j)^2 \leq Nt^2$$

with some $t \geq 1$.

Rudelson's theorem

Then for every $\epsilon > 0$ there exists a set $J \subset \{1, \dots, M\}$ of indices with cardinality

$$m := |J| \le C \frac{t^2}{\epsilon^2} N \log \frac{Nt^2}{\epsilon^2}$$

such that for any $f = \sum_{i=1}^{N} c_i u_i$ we have

$$(1-\epsilon)\|f\|_2^2 \le \frac{1}{m} \sum_{j \in J} f(x^j)^2 \le (1+\epsilon)\|f\|_2^2.$$

A slight improvement

Theorem (VT, 2017)

Let $\{u_i\}_{i=1}^N$ *be an orthonormal system, satisfying condition* **E**. *Then for every* $\epsilon > 0$ *there exists a set* $\{\xi^j\}_{j=1}^m \subset \Omega$ *with*

$$m \leq C\frac{t^2}{\epsilon^2} N \log N$$

such that for any $f = \sum_{i=1}^N c_i u_i$ *we have*

$$(1-\epsilon)\|f\|_2^2 \leq \frac{1}{m}\sum_{j=1}^m f(\xi^j)^2 \leq (1+\epsilon)\|f\|_2^2.$$

We now comment on a recent breakthrough result by J. Batson, D.A. Spielman, and N. Srivastava, 2012. We formulate their result in our notations. Let as above $\Omega_M=\{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu(x^j)=1/M$, $j=1,\dots,M$. Assume that $\{u_i(x)\}_{i=1}^N$ is a real orthonormal on Ω_M system. Then for any number $d>1$ there exist a set of weights $w_j\ge 0$ such that $|\{j: w_j\neq 0\}|\le dN$ so that for any $f\in \operatorname{span}\{u_1,\dots,u_N\}$ we have

$$\|f\|_2^2 \le \sum_{j=1}^M w_j f(x^j)^2 \le \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}\|f\|_2^2.$$

A comment

The proof of this result is based on a delicate study of the m-term approximation of the identity matrix I with respect to the system $\mathcal{D} := \{G(x)\}_{x\in\Omega}$, $G(x) := [u_i(x)u_j(x)]_{i,j=1}^N$ in the spectral norm. The authors control the change of the maximal and minimal eigenvalues of a matrix, when they add a rank one matrix of the form $wG(x)$. Their proof provides an algorithm for construction of the weights $\{w_j\}$. In particular, this implies that

$$X_N(\Omega_M) \in \mathcal{M}^w(m,2,\epsilon) \quad \text{provided} \quad m \geq CN\epsilon^{-2}$$

with large enough C.

Definition of the entropy numbers

Let X be a Banach space and let B_X denote the unit ball of X with the center at 0. Denote by $B_X(y,r)$ a ball with center y and radius r: $\{x \in X : \|x-y\| \le r\}$. For a compact set A and a positive number ε we define the covering number $N_\varepsilon(A,X)$ as follows

$$N_\varepsilon(A,X) := \min\{n : \exists y^1,\dots,y^n, y^j \in A : A \subseteq \cup_{j=1}^n B_X(y^j,\varepsilon)\}.$$

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It is convenient to consider along with the entropy $H_\varepsilon(A,X) := \log_2 N_\varepsilon(A,X)$ the entropy numbers $\varepsilon_k(A,X)$:

$$\varepsilon_k(A,X) := \inf\{\varepsilon : \exists y^1,\dots,y^{2^k} \in A : A \subseteq \cup_{j=1}^{2^k} B_X(y^j,\varepsilon)\}.$$

In our definition of $N_\varepsilon(A,X)$ and $\varepsilon_k(A,X)$ we require $y^j \in A$. In a standard definition of $N_\varepsilon(A,X)$ and $\varepsilon_k(A,X)$ this restriction is not imposed. However, it is well known that these characteristics may differ at most by a factor 2.

Conditional theorem

Theorem (4; VT2017)

Suppose that a real N-dimensional subspace X_N satisfies the following condition on the entropy numbers of the unit ball $X^1_N := \{f \in X_N : \|f\|_1 \le 1\}$ with $B \ge 1$

$$\varepsilon_k(X^1_N, L_\infty) \le B \begin{cases} N/k, & k \le N, \\ 2^{-k/N}, & k \ge N. \end{cases}$$

Then there exists a set of $m \le C_1 NB(\log_2(2N\log_2(8B)))^2$ points $\xi^j \in \Omega$, $j = 1, \dots, m$, with large enough absolute constant C_1, such that for any $f \in X_N$ we have

$$\frac{1}{2}\|f\|_1 \le \frac{1}{m}\sum_{j=1}^{m} |f(\xi^j)| \le \frac{3}{2}\|f\|_1.$$

Concentration measure lemma

The following lemma is from J. Bourgain, J. Lindenstrauss and V. Milman, 1989.

Lemma (BLM, 1989)

Let $\{g_j\}_{j=1}^m$ *be independent random variables with* $\mathbb{E}g_j = 0$, $j = 1, \dots, m$, *which satisfy*

$$\|g_j\|_1 \le 2, \qquad \|g_j\|_\infty \le M, \qquad j = 1, \dots, m.$$

Then for any $\eta \in (0,1)$ *we have the following bound on the probability*

$$\mathbb{P}\left\{\left|\sum_{j=1}^m g_j\right| \ge m\eta\right\} < 2\exp\left(-\frac{m\eta^2}{8M}\right).$$

Notation

We now consider measurable functions $f(\mathbf{x})$, $\mathbf{x} \in \Omega$. For $1 \le q < \infty$ define

$$L_{\mathbf{z}}^q(f) := \frac{1}{m} \sum_{j=1}^m |f(\mathbf{x}^j)|^q - \|f\|_q^q, \qquad \mathbf{z} := (\mathbf{x}^1, \dots, \mathbf{x}^m).$$

Let μ be a probabilistic measure on Ω. Denote $\mu^m := \mu \times \cdots \times \mu$ the probabilistic measure on $\Omega^m := \Omega \times \cdots \times \Omega$. We need the following inequality, which is a corollary of the above Lemma.

Key inequality

Proposition (VT, 2017)

Let $f_j \in L_1(\Omega)$ *be such that*

$$\|f_j\|_1 \le 1/2, \quad j = 1, 2; \qquad \|f_1 - f_2\|_\infty \le \delta.$$

Then

$$\mu^m\{\mathbf{z} : |L^1_{\mathbf{z}}(f_1) - L^1_{\mathbf{z}}(f_2)| \ge \eta\} < 2\exp\left(-\frac{m\eta^2}{16\delta}\right). \qquad (5)$$

Some more notations

We consider the case X is $\mathcal{C}(\Omega)$ the space of functions continuous on a compact subset Ω of $\mathbb{R}^d$ with the norm

$$\|f\|_\infty := \sup_{\mathbf{x}\in\Omega} |f(\mathbf{x})|.$$

We use the abbreviated notations

$$\varepsilon_n(W) := \varepsilon_n(W, \mathcal{C}).$$

In our case

$$W := W(Q) := \{t \in \mathcal{T}(Q) : \|t\|_1 = 1/2\}. \tag{6}$$

The entropy bound

Theorem (6; VT, 2017)

For any $Q \subset \Pi(\mathbf{N})$ *with* $\mathbf{N} = (2^n, \dots, 2^n)$ *we have*

$$\varepsilon_k(\mathcal{T}(Q)_1, L_\infty) \le 2\varepsilon_k := 2C_4(d) \begin{cases} n^{3/2}(|Q|/k), & k \le 2|Q|, \\ n^{3/2}2^{-k/(2|Q|)}, & k \ge 2|Q|. \end{cases}$$

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Specify $\eta = 1/4$. Denote $\delta_j := \varepsilon_{2^j}$, $j = 0, 1, \dots$, and consider minimal δ_j-nets $\mathcal{N}_j \subset W$ of W in $\mathcal{C}(\mathbb{T}^d)$. We use the notation $N_j := |\mathcal{N}_j|$. Let J be the minimal j satisfying $\delta_j \le 1/16$. For $j = 1, \dots, J$ we define a mapping A_j that associates with a function $f \in W$ a function $A_j(f) \in \mathcal{N}_j$ closest to f in the $\mathcal{C}$ norm.

Building a chain

Then, clearly,

$$\|f - A_j(f)\|_{\mathcal{C}} \le \delta_j.$$

We use the mappings A_j, $j = 1, \dots, J$ to associate with a function $f \in W$ a sequence (a chain) of functions $f_J, f_{J-1}, \dots, f_1$ in the following way

$$f_J := A_J(f), \quad f_j := A_j(f_{j+1}), \quad j = J-1, \dots, 1.$$

Reduction to simple events

Set

$$\eta_j := \frac{1}{16nd}, \quad j = 1, \dots, J.$$

Rewriting

$$L^1_{\mathbf{z}}(f_J) = L^1_{\mathbf{z}}(f_J) - L^1_{\mathbf{z}}(f_{J-1}) + \cdots + L^1_{\mathbf{z}}(f_2) - L^1_{\mathbf{z}}(f_1) + L^1_{\mathbf{z}}(f_1)$$

we conclude that if $|L^1_{\mathbf{z}}(f)| \geq 1/4$ then at least one of the following events occurs:

$$|L^1_{\mathbf{z}}(f_j) - L^1_{\mathbf{z}}(f_{j-1})| \geq \eta_j \quad \text{for some} \quad j \in (1, J] \quad \text{or} \quad |L^1_{\mathbf{z}}(f_1)| \geq \eta_1.$$

In the rest of the proof we use the Proposition to estimate accurately the probability of the above events.