Sampling discretization of integral norms. Lecture 2

Vladimir Temlyakov

Chemnitz, September, 2019

Universal discretization problem

Let $\mathcal{X}_N := \{X_N^j\}_{j=1}^k$ be a collection of linear subspaces X_N^j of the $L_q(\Omega)$, $1 \le q \le \infty$. We say that a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ provides universal discretization for the collection $\mathcal{X}_N$ if, Let $\mathcal{X}_N := \{X_N^j\}_{j=1}^k$ be a collection of linear subspaces X_N^j of the $L_q(\Omega)$, $1 \le q \le \infty$. We say that a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ provides universal discretization for the collection $\mathcal{X}_N$ if, in the case $1 \le q < \infty$, there are two positive constants $C_i(d,q)$, $i = 1, 2$, such that for each $j \in [1, k]$ and any $f \in X_N^j$ we have

$$C_1(d,q)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^m |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \qquad (1)$$

Universal discretization problem

Let $\mathcal{X}_N := \{X_N^j\}_{j=1}^k$ be a collection of linear subspaces X_N^j of the $L_q(\Omega)$, $1 \le q \le \infty$. We say that a set $\{\xi^\nu \in \Omega, \nu = 1, \dots, m\}$ provides universal discretization for the collection $\mathcal{X}_N$ if, in the case $1 \le q < \infty$, there are two positive constants $C_i(d,q)$, $i = 1, 2$, such that for each $j \in [1,k]$ and any $f \in X_N^j$ we have

$$C_1(d,q)\|f\|_q^q \le \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^q \le C_2(d,q)\|f\|_q^q. \tag{1}$$

In the case $q = \infty$ for each $j \in [1,k]$ and any $f \in X_N^j$ we have

$$C_1(d)\|f\|_\infty \le \max_{1\le\nu\le m} |f(\xi^\nu)| \le \|f\|_\infty. \tag{2}$$

Main new result

We are primarily interested in the Universal discretization for the collection of subspaces of trigonometric polynomials with frequencies from parallelepipeds (rectangles). For $\mathbf{s} \in \mathbb{Z}_+^d$ define

$$R(\mathbf{s}) := \{\mathbf{k} \in \mathbb{Z}^d : |k_j| < 2^{s_j}, \quad j = 1, \dots, d\}.$$

Clearly, $R(\mathbf{s}) = \Pi(\mathbf{N})$ with $N_j = 2^{s_j} - 1$. Consider the collection $\mathcal{C}(n,d) := \{\mathcal{T}(R(\mathbf{s})), \|\mathbf{s}\|_1 = n\}$.

Main new result

We are primarily interested in the Universal discretization for the collection of subspaces of trigonometric polynomials with frequencies from parallelepipeds (rectangles). For $\mathbf{s} \in \mathbb{Z}_+^d$ define

$$R(\mathbf{s}) := \{\mathbf{k} \in \mathbb{Z}^d : |k_j| < 2^{s_j}, \quad j = 1, \dots, d\}.$$

Clearly, $R(\mathbf{s}) = \Pi(\mathbf{N})$ with $N_j = 2^{s_j} - 1$. Consider the collection $\mathcal{C}(n,d) := \{\mathcal{T}(R(\mathbf{s})), \|\mathbf{s}\|_1 = n\}$. The following result is obtained by VT, 2017.

Theorem (1; VT, 2017)

For every $1 \le q \le \infty$ there exists a large enough constant $C(d,q)$, which depends only on d and q, such that for any $n \in \mathbb{N}$ there is a set $\Xi_m := \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$, with $m \le C(d,q)2^n$ that provides universal discretization in L_q for the collection $\mathcal{C}(n,d)$.

Dispersion

Let $d \geq 2$ and $[0,1)^d$ be the d-dimensional unit cube. For $\mathbf{x}, \mathbf{y} \in [0,1)^d$ with $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$ we write $\mathbf{x} < \mathbf{y}$ if this inequality holds coordinate-wise.

Dispersion

Let $d \geq 2$ and $[0,1)^d$ be the d-dimensional unit cube. For $\mathbf{x}, \mathbf{y} \in [0,1)^d$ with $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$ we write $\mathbf{x} < \mathbf{y}$ if this inequality holds coordinate-wise. For $\mathbf{x} < \mathbf{y}$ we write $[\mathbf{x}, \mathbf{y})$ for the axis-parallel box $[x_1, y_1) \times \cdots \times [x_d, y_d)$ and define

$$\mathcal{B} := \{[\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in [0,1)^d, \mathbf{x} < \mathbf{y}\}.$$

Let $d \geq 2$ and $[0,1)^d$ be the d-dimensional unit cube. For $\mathbf{x}, \mathbf{y} \in [0,1)^d$ with $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$ we write $\mathbf{x} < \mathbf{y}$ if this inequality holds coordinate-wise. For $\mathbf{x} < \mathbf{y}$ we write $[\mathbf{x}, \mathbf{y})$ for the axis-parallel box $[x_1, y_1) \times \cdots \times [x_d, y_d)$ and define

$$\mathcal{B} := \{[\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in [0,1)^d, \mathbf{x} < \mathbf{y}\}.$$

For $n \geq 1$ let T be a set of points in $[0,1)^d$ of cardinality $|T| = n$. The volume of the largest empty (from points of T) axis-parallel box, which can be inscribed in $[0,1)^d$, is called the dispersion of T:

$$\operatorname{disp}(T) := \sup_{B \in \mathcal{B} : B \cap T = \emptyset} vol(B).$$

A bound on the minimal dispersion

An interesting extremal problem is to find (estimate) the minimal dispersion of point sets of fixed cardinality:

$$\mathrm{disp}^*(n,d) := \inf_{T\subset[0,1)^d, |T|=n} \mathrm{disp}(T).$$

A bound on the minimal dispersion

An interesting extremal problem is to find (estimate) the minimal dispersion of point sets of fixed cardinality:

$$\text{disp*}(n,d) := \inf_{T\subset[0,1)^d, |T|=n} \text{disp}(T).$$

It is known that

$$\text{disp*}(n,d) \le C^*(d)/n. \tag{6}$$

A bound on the minimal dispersion

An interesting extremal problem is to find (estimate) the minimal dispersion of point sets of fixed cardinality:

$$\text{disp*}(n,d) := \inf_{T\subset[0,1)^d, |T|=n} \text{disp}(T).$$

It is known that

$$\text{disp*}(n,d) \le C^*(d)/n. \tag{6}$$

Inequality (6) with $C^*(d) = 2^{d-1}\prod_{i=1}^{d-1} p_i$, where p_i denotes the ith prime number, was proved by A. Dumitrescu and M. Jiang, 2013 (see also G. Rote and F. Tichy, 1996).

A bound on the minimal dispersion

An interesting extremal problem is to find (estimate) the minimal dispersion of point sets of fixed cardinality:

$$\mathrm{disp}^*(n,d) := \inf_{T\subset[0,1)^d, |T|=n} \mathrm{disp}(T).$$

It is known that

$$\mathrm{disp}^*(n,d) \le C^*(d)/n. \tag{6}$$

Inequality (6) with $C^*(d) = 2^{d-1}\prod_{i=1}^{d-1} p_i$, where p_i denotes the ith prime number, was proved by A. Dumitrescu and M. Jiang, 2013 (see also G. Rote and F. Tichy, 1996). A. Dumitrescu and M. Jiang used the Halton-Hammersly set of n points.

A bound on the minimal dispersion

An interesting extremal problem is to find (estimate) the minimal dispersion of point sets of fixed cardinality:

$$\mathrm{disp}^*(n,d) := \inf_{T\subset[0,1)^d, |T|=n} \mathrm{disp}(T).$$

It is known that

$$\mathrm{disp}^*(n,d) \le C^*(d)/n. \tag{6}$$

Inequality (6) with $C^*(d) = 2^{d-1}\prod_{i=1}^{d-1} p_i$, where p_i denotes the ith prime number, was proved by A. Dumitrescu and M. Jiang, 2013 (see also G. Rote and F. Tichy, 1996). A. Dumitrescu and M. Jiang used the Halton-Hammersly set of n points. Inequality (6) with $C^*(d) = 2^{7d+1}$ was proved by C. Aistleitner, A. Hinrichs, and D. Rudolf, 2017.

Difinition of the (t, r, d)-net

C. Aistleitner, A. Hinrichs, and D. Rudolf, following G. Larcher, used the (t, r, d)-nets.

Definition

A (t, r, d)-net (in base 2) is a set T of 2^r points in $[0, 1)^d$ such that each dyadic box $[(a_1 - 1)2^{-s_1}, a_1 2^{-s_1}) \times \cdots \times [(a_d - 1)2^{-s_d}, a_d 2^{-s_d})$, $1 \le a_j \le 2^{s_j}$, $j = 1, \dots, d$, of volume 2^{t-r} contains exactly 2^t points of T.

Universal discretization in L_∞

Theorem (2; VT, 2017)

Let a set T with cardinality $|T| = 2^r =: m$ have dispersion satisfying the bound $disp(T) < C(d)2^{-r}$ with some constant $C(d)$. Then there exists a constant $c(d) \in \mathbb{N}$ such that the set $2\pi T := \{2\pi \mathbf{x} : \mathbf{x} \in T\}$ provides the universal discretization in L_∞ for the collection $\mathcal{C}(n,d)$ with $n = r - c(d)$.

Universal discretization in L_∞

Theorem (2; VT, 2017)

Let a set T with cardinality $|T| = 2^r =: m$ have dispersion satisfying the bound $disp(T) < C(d)2^{-r}$ with some constant $C(d)$. Then there exists a constant $c(d) \in \mathbb{N}$ such that the set $2\pi T := \{2\pi \mathbf{x} : \mathbf{x} \in T\}$ provides the universal discretization in L_∞ for the collection $\mathcal{C}(n,d)$ with $n = r - c(d)$.

Theorem (3; VT, 2017)

Assume that $T \subset [0,1)^d$ is such that the set $2\pi T$ provides universal discretization in L_∞ for the collection $\mathcal{C}(n,d)$. Then there exists a positive constant $C(d)$ with the following property $disp(T) \le C(d)2^{-n}$.

Dirichlet kernel

We need some classical trigonometric polynomials. We begin with the univariate case. The Dirichlet kernel of order n:

$$\mathcal{D}_n(x) := \sum_{|k|\le n} e^{ikx} = e^{-inx}(e^{i(2n+1)x} - 1)(e^{ix} - 1)^{-1}$$

$$= \big(\sin(n + 1/2)x\big) \; / \; \sin(x/2)$$

is an even trigonometric polynomial.

The de la Vallée Poussin kernel:

$$\mathcal{V}_n(x) := n^{-1} \sum_{l=n}^{2n-1} \mathcal{D}_l(x),$$

is an even trigonometric polynomial of order $2n-1$ with the majorant

$$\left|\mathcal{V}_n(x)\right| \le C \min\left(n,\ (nx^2)^{-1}\right), \quad |x| \le \pi. \tag{7}$$

Simple lemma

The above relation (7) easily implies the following lemma.

Lemma (1; VT, 2017)

For a set $\Xi_m := \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}$ *satisfying the condition* $|\Xi_m \cap [x(l-1), x(l))| \le b$, $x(l) := \pi l/2n$, $l = 1, \dots, 4n$, *we have*

$$\sum_{\nu=1}^{m} \big|\mathcal{V}_n(x - \xi^\nu)\big| \le Cbn.$$

We use the above Lemma (1) to prove a one-sided inequality.

Bounds for the operator norm

Lemma (2; VT, 2017)

For a set $\Xi_m := \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}$ *satisfying the condition* $|\Xi_m \cap [x(l-1), x(l))| \leq b$, $x(l) := \pi l/2n$, $l = 1, \dots, 4n$, *we have for* $1 \leq q \leq \infty$

$$\left\| m^{-1} \sum_{\nu=1}^{m} a_\nu \mathcal{V}_n(x - \xi^\nu) \right\|_q \leq C(bn/m)^{1-1/q} \left(\frac{1}{m} \sum_{\nu=1}^{m} |a_\nu|^q \right)^{1/q}.$$

Multivariate kernel

We now proceed to the multivariate case. Denote the multivariate de la Vallée Poussin kernels:

$$\mathcal{V}_{\mathbf{N}}(\mathbf{x}) := \prod_{j=1}^{d} \mathcal{V}_{N_j}(x_j), \qquad \mathbf{N} = (N_1, \dots, N_d).$$

In the same way as above in the univariate case one can establish the following multivariate analog of Lemma (2).

Lemma (3; VT, 2017)

For a set $\Xi_m := \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$ *satisfying the condition* $|\Xi_m \cap [\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n}+\mathbf{1}))| \le b$, $\mathbf{n} \in P'(\mathbf{N})$, $\mathbf{1}$ *is a vector with coordinates* 1 *for all* j, *we have for* $1 \le q \le \infty$

$$\left\| \frac{1}{m} \sum_{\nu=1}^m a_\nu \mathcal{V}_{\mathbf{N}}(\mathbf{x} - \xi^\nu) \right\|_q \le C(d)(bv(\mathbf{N})/m)^{1-1/q} \left(\frac{1}{m} \sum_{\nu=1}^m |a_\nu|^q \right)^{1/q}.$$

Upper bound for the discrete norm

Theorem (4; VT, 2017)

Let a set $\Xi_m := \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$ *satisfy the condition* $|\Xi_m \cap [\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n}+\mathbf{1}))| \le b(d)$, $\mathbf{n} \in P'(\mathbf{N})$, $\mathbf{1}$ *is a vector with coordinates* 1 *for all* j. *Then for* $m \ge v(\mathbf{N})$ *we have for each* $f \in \mathcal{T}(\mathbf{N})$ *and* $1 \le q \le \infty$

$$\left(\frac{1}{m}\sum_{\nu=1}^m |f(\xi^\nu)|^q\right)^{1/q} \le C(d)\|f\|_q.$$

Property E(b)

We now proceed to the inverse bounds for the discrete norm. Denote

$$\Delta(\mathbf{n}) := [\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n}+\mathbf{1})), \quad \mathbf{n} \in P'(\mathbf{N}).$$

Suppose that a sequence $\Xi_m := \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$ has the following property.

Property E(b). There is a number $b \in \mathbb{N}$ such that for any $\mathbf{n} \in P'(\mathbf{N})$ we have

$$|\Delta(\mathbf{n}) \cap \Xi_m| = b.$$

Clearly, in this case $m = v(\mathbf{N})b$, where $v(\mathbf{N}) = |P'(\mathbf{N})|$.

Main lemma for the lower bound

Lemma (4; VT, 2017)

Suppose that two sequences $\Xi_m := \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$ *and* $\Gamma_m := \{\gamma^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$ *satisfy the following condition. For a given* $j \in \{1, \dots, d\}$, γ^ν *may only differ from* ξ^ν *in the* j*th coordinate. Moreover, assume that if* $\xi^\nu \in \Delta(\mathbf{n})$ *then also* $\gamma^\nu \in \Delta(\mathbf{n})$. *Finally, assume that* Ξ_m *has property* $E(b)$ *with* $b \le C'(d)$. *Then for* $f \in \mathcal{T}(\mathbf{K})$ *with* $\mathbf{K} \le \mathbf{N}$ *we have*

$$\frac{1}{m}\sum_{\nu=1}^m \left| |f(\xi^\nu)|^q - |f(\gamma^\nu)|^q \right| \le C(d,q)(K_j/N_j)\|f\|_q^q.$$

Arbitrary trigonometric polynomials

For $n \in \mathbb{N}$ denote $\Pi_n := \Pi(\mathbf{N}) \cap \mathbb{Z}^d$ with $\mathbf{N} = (2^{n-1} - 1, \dots, 2^{n-1} - 1)$, where, as above, $\Pi(\mathbf{N}) := [-N_1, N_1] \times \cdots \times [-N_d, N_d]$. Then $|\Pi_n| = (2^n - 1)^d < 2^{dn}$. Let $v \in \mathbb{N}$ and $v \leq |\Pi_n|$. Consider

$$\mathcal{S}(v, n) := \{Q \subset \Pi_n : |Q| = v\}.$$

Arbitrary trigonometric polynomials

For $n \in \mathbb{N}$ denote $\Pi_n := \Pi(\mathbf{N}) \cap \mathbb{Z}^d$ with $\mathbf{N} = (2^{n-1} - 1, \dots, 2^{n-1} - 1)$, where, as above, $\Pi(\mathbf{N}) := [-N_1, N_1] \times \dots \times [-N_d, N_d]$. Then $|\Pi_n| = (2^n - 1)^d < 2^{dn}$. Let $v \in \mathbb{N}$ and $v \leq |\Pi_n|$. Consider

$$\mathcal{S}(v, n) := \{Q \subset \Pi_n : |Q| = v\}.$$

Then it is easy to see that

$$|\mathcal{S}(v, n)| = \binom{|\Pi_n|}{v} < 2^{dnv}.$$

Universal discretization problem

We are interested in solving the following problem of universal discretization. For a given $\mathcal{S}(v,n)$ and $q\in[1,\infty)$ find a condition on m such that there exists a set $\xi=\{\xi^\nu\}_{\nu=1}^m$ with the property: for any $Q\in\mathcal{S}(v,n)$ and each $f\in\mathcal{T}(Q)$ we have

$$C_1(q,d)\|f\|_q^q\le\frac{1}{m}\sum_{\nu=1}^m|f(\xi^\nu)|^q\le C_2(q,d)\|f\|_q^q.$$

Universal discretization problem

We are interested in solving the following problem of universal discretization. For a given $\mathcal{S}(v,n)$ and $q\in[1,\infty)$ find a condition on m such that there exists a set $\xi=\{\xi^\nu\}_{\nu=1}^m$ with the property: for any $Q\in\mathcal{S}(v,n)$ and each $f\in\mathcal{T}(Q)$ we have

$$C_1(q,d)\|f\|_q^q\le \frac{1}{m}\sum_{\nu=1}^m|f(\xi^\nu)|^q\le C_2(q,d)\|f\|_q^q.$$

We present results for $q=2$ and $q=1$.

The case $q = 2$

We begin with a general construction. Let $X_N = \text{span}(u_1, \dots, u_N)$, where $\{u_j\}_{j=1}^N$ is a real orthonormal system on $\mathbb{T}^d$. With each $\mathbf{x} \in \mathbb{T}^d$ we associate the matrix $G(\mathbf{x}) := [u_i(\mathbf{x})u_j(\mathbf{x})]_{i,j=1}^N$. Clearly, $G(\mathbf{x})$ is a symmetric matrix. For a set of points $\xi^k \in \mathbb{T}^d$, $k = 1, \dots, m$, and $f = \sum_{i=1}^N b_i u_i$ we have

$$\frac{1}{m}\sum_{k=1}^m f(\xi^k)^2 - \int_{\mathbb{T}^d} f(x)^2 d\mu = \mathbf{b}^T \left(\frac{1}{m}\sum_{k=1}^m G(\xi^k) - I \right) \mathbf{b},$$

where $\mathbf{b} = (b_1, \dots, b_N)^T$ is the column vector. Therefore,

$$\left| \frac{1}{m}\sum_{k=1}^m f(\xi^k)^2 - \int_{\mathbb{T}^d} f(x)^2 d\mu \right| \le \left\| \frac{1}{m}\sum_{k=1}^m G(\xi^k) - I \right\| \|\mathbf{b}\|_2^2.$$

Probability bound

We recall that the system $\{u_j\}_{j=1}^N$ satisfies Condition **E** if there exists a constant t such that

$$w(x) := \sum_{i=1}^{N} u_i(x)^2 \leq Nt^2.$$

Probability bound

We recall that the system $\{u_j\}_{j=1}^N$ satisfies Condition **E** if there exists a constant t such that

$$w(x) := \sum_{i=1}^{N} u_i(x)^2 \le Nt^2.$$

Let points $\mathbf{x}^k$, $k = 1, \dots, m$, be independent uniformly distributed on $\mathbb{T}^d$ random variables. Then with a help of deep results on random matrices it was proved that

$$\mathbb{P}\left\{\left\|\sum_{k=1}^{m}(G(\mathbf{x}^k) - I)\right\| \ge m\eta\right\} \le N \exp\left(-\frac{m\eta^2}{ct^2 N}\right)$$

with an absolute constant c.

The union bound

Consider real trigonometric polynomials from the collection $\mathcal{S}(v,n)$. Using the union bound for the probability we get that the probability of the event

$$\left\|\sum_{k=1}^{m}(G_Q(\mathbf{x}^k)-I)\right\| \le m\eta \quad \text{for all} \quad Q\in\mathcal{S}(v,n)$$

is bounded from below by

$$1-|\mathcal{S}(v,n)|v\exp\left(-\frac{m\eta^2}{cv}\right).$$

For any fixed $\eta\in(0,1/2]$ the above number is positive provided $m\ge C(d)\eta^{-2}v^2n$ with large enough $C(d)$. The above argument proves the following result.

Main result for $q = 2$

Theorem (Dai, Prymak, VT, Tikhonov, 2018.)

There exist three positive constants $C_i(d)$, $i = 1, 2, 3$, *such that for any* $n, v \in \mathbb{N}$ *and* $v \le |\Pi_n|$ *there is a set* $\xi = \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$, *with* $m \le C_1(d) v^2 n$, *which provides universal discretization in* L_2 *for the collection* $\mathcal{S}(v, n)$: *for any* $f \in \cup_{Q \in \mathcal{S}(v,n)} \mathcal{T}(Q)$

$$C_2(d)\|f\|_2^2 \le \frac{1}{m}\sum_{\nu=1}^{m} |f(\xi^\nu)|^2 \le C_3(d)\|f\|_2^2.$$

Similar to the case $q = 2$ a result on the universal discretization for the collection $\mathcal{S}(v, n)$ will be derived from the probabilistic result on the Marcinkiewicz-type theorem for $\mathcal{T}(Q)$, $Q \subset \Pi_n$. However, the probabilistic technique used in the case of $q = 1$ is different from the probabilistic technique used in the case $q = 2$. The proof from VT, 2017, gives the following result.

Probability bound

Theorem (VT, 2017)

Let points $\mathbf{x}^j \in \mathbb{T}^d$, $j = 1, \dots, m$, *be independently and uniformly distributed on* $\mathbb{T}^d$. *There exist positive constants* $C_1(d)$, C_2, C_3, *and* $\kappa \in (0,1)$ *such that for any* $Q \subset \Pi_n$ *and* $m \geq y C_1(d) |Q| n^{7/2}$, $y \geq 1$,

$$\mathbb{P}\left\{\forall f \in \mathcal{T}(Q), \quad C_2\|f\|_1 \leq \frac{1}{m}\sum_{j=1}^{m} |f(\mathbf{x}^j)| \leq C_3\|f\|_1\right\} \geq 1 - \kappa^y.$$

Therefore, using the union bound for probability we obtain the Marcinkiewicz-type inequalities for all $Q \in \mathcal{S}(v,n)$ with probability at least $1 - |\mathcal{S}(v,n)|\kappa^y$. Choosing $y = y(v,n) := C(d)vn$ with large enough $C(d)$ we get

$$1 - |\mathcal{S}(v,n)|\kappa^{y(v,n)} > 0.$$

This argument implies the following result on universality in L_1.

Main result for $q = 1$

Theorem (Dai, Prymak, VT, Tikhonov, 2018.)

There exist three positive constants $C_1(d)$, C_2, C_3, *such that for any* $n, v \in \mathbb{N}$ *and* $v \leq |\Pi_n|$ *there is a set* $\xi = \{\xi^\nu\}_{\nu=1}^m \subset \mathbb{T}^d$, *with* $m \leq C_1(d) v^2 n^{9/2}$, *which provides universal discretization in* L_1 *for the collection* $\mathcal{S}(v, n)$: *for any* $f \in \cup_{Q \in \mathcal{S}(v,n)} \mathcal{T}(Q)$

$$C_2 \|f\|_1 \leq \frac{1}{m} \sum_{\nu=1}^m |f(\xi^\nu)| \leq C_3 \|f\|_1.$$