Random wavelet series - Part I

Béatrice VEDEL

Université de Bretagne Sud

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Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space. Let $I \subset \mathbb{R}$ be an interval.

Definition (stochastic process)

A continuous-time stochastic process $\{X_t \, : \, t \in I\}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ is a family of random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}$ and indexed by I.

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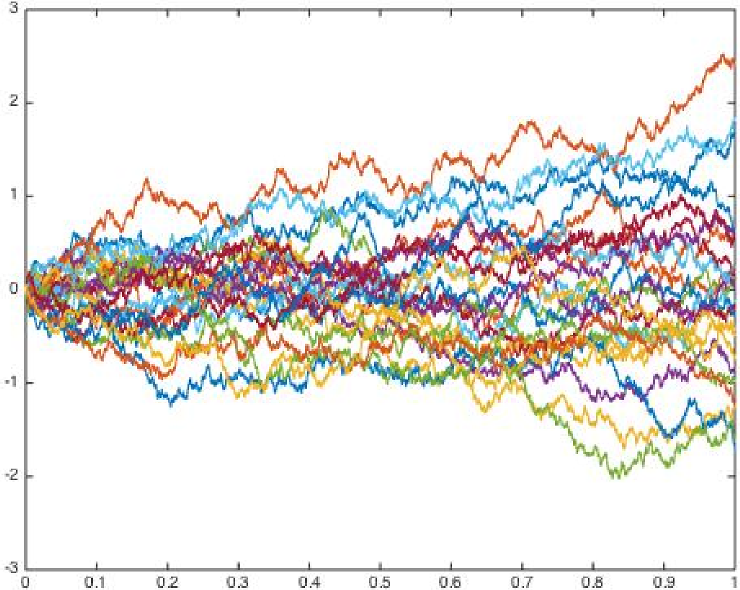
- For all $\omega \in \Omega$, $X(\cdot, \omega) \,:\, t \in I$ is a real-valued function, called a **sample path**. The study of the properties of the sample paths of the stochastic process is an important part of the theory of stochastic processes.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Definition (Finite dimensional laws)

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They are characterized by the family of **cumulative distribution functions** $\{F^X_{t_1,..,t_n} \, : \, n \in \mathbb{N},\ t_1, ..., t_n \in I\}$ given by

$$F^X_{t_1,...,t_n}(x_1, ..., x_n) = \mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2, ..., X_{t_n} \leq x_n)$$

for all $n \in \mathbb{N}$ and $t_1, ..., t_n \in I$.

Definition (Equality of processes.)

1. The two stochastic processes X and Y are said to be **equal in the sense of finite-dimensional distributions (f.d.d.)** if for all $n \in \mathbb{N}$ and $t_1, ..., t_n \in I$,

$$F^X(t_1, ..., t_n) = F^Y(t_1, ..., t_n)$$

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2. We say that X and Y are **modifications or versions** of each other if they are defined on the same probabily space and if

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3. We say that X and Y are **indistinguishable** if they are defined on the same probability space Ω and if there exists $N \subset \Omega$ neglectable such that

$$X(t, \omega) = Y(t, \omega), \quad \forall t \in I \text{ and } \omega \in \Omega \setminus N.$$

Proposition

$$X \text{ and } Y \textbf{ indistinguishable} \Rightarrow X \text{ and } Y \text{ are } \textbf{versions of each other}$$
$$\Rightarrow X \text{ and } Y \text{ have same f.d.d.}$$

Let us mention also the existence of stochastic processes for consistent finite dimensional distributions. A family of f.d.d. $\{\mathbb{P}_{t_1,\dots,t_n},\, n \in \mathbb{N}, t_1, ..., t_n \in I\}$ is said to be consistent if

1. For all permutation π,

$$\mathbb{P}_{t_{\pi(1)},\dots,t_{\pi(n)})}[A_{\pi(1)} \times ... \times A_{\pi(n)}] = \mathbb{P}_{t_1,\dots,t_n}[A_1 \times ... \times A_n]$$

   for all $A_1, ..., A_n \in \mathcal{A}$,

2. For all $t_{n+1} \in I$,

$$\mathbb{P}_{t_1,\dots,t_{n+1}}[A \times \mathbb{R}] = \mathbb{P}_{t_1,\dots,t_n}[A]$$

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Theorem (Kolmogorov consistency theorem)

Let us consider for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in I$ a probability measure $\mathbb{P}_{t_1,\dots,t_n}$. Suppose that the collection $\mathcal{Q} = \{\mathbb{P}_{t_1,\dots,t_n},\ n \in \mathbb{N},\ t_1, \dots, t_n \in I\}$ is consistent. Then, there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a stochastic process X on $(\Omega, \mathcal{A}, \mathbb{P})$ such that X admits Q as its finite dimensional distributions.

A major result is given by the following theorem

Theorem (Kolmogorov-Chenstov theorem)

Let X a stochastic process indexed by $t \in I$, with I an interval. If there exists $\alpha, \beta > 0$ and a constant $C > 0$ such that

$$\mathbb{E}[|X_t - X_s|^\alpha] \leq C|t-s|^{1+\beta}$$

for all $s, t \in I$, then there exists a version $\widetilde{X}$ of X whose sample paths are locally γ-hölderian for all $\gamma \in]O, \beta/\alpha[$ that is :
For all $\omega \in \Omega$, for all compact $K \subset I$, there exists a constant $C_\omega(K)$ such that

$$|\widetilde{X}(t) - \widetilde{X}(s)| \leq C_\omega(K)|t-s|^\gamma, \quad \forall t, s \in K.$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space.

Lemma (Borel-Cantelli lemma)

If the sequence of events $(A_n)_{n\in\mathbb{N}}$ satisfies

$$\sum \mathbb{P}(A_n) < +\infty$$

then

$$\mathbb{P}(\limsup A_n) = \mathbb{P}(\bigcap_n \bigcup_{k\geq n} A_k) = 0,$$

which means that the probability that infinitely many of the events A_n occur is null.

Definition

A stochastic process $\{X(t),\, t \in I\}$ is **gaussian** if all its finite dimensional laws are gaussian, or equivalently, iff for all $p \in \mathbb{N}$, $a_1, ...a_p \in \mathbb{R}$ and $t_1, .., t_p \in I$

$$a_1 X(t_1) + ... + a_p X(t_p)$$

- They are characterized by $m(t) = \mathbb{E}\, X(t)$ and the covariance operator $K(t,s) = Cov(X(t), X(s))$
- K is symmetric and $\forall p \in \mathbb{N},\ (t_1, ..., t_p) \in I^p$, the matrix $(K(t_i, t_j))_{1 \leq i,j \leq p}$ is positive semi-definite.
- For any m and for such K, there exists a gaussian process of mean m and of covariance operator K

Proposition (Gaussian Kolmogorov-Centsov)

Let X a centered gaussian process with covariance operator K. If there exists $\epsilon > 0$ and $C > 0$ such that

$$K(t,t) + K(s,s) - 2K(t,s) \leq C|t-s|^{\epsilon}$$

for all $t, s \in I$, then there exists a version $\widetilde{X}$ of X whose sample paths are locally Hölder of exponent γ for all $\gamma \in]0, \epsilon/2[$.

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- 1926 - Kolmogorov : There exists a function in L^1 such that its Fourier series diverges almost everywhere
- 1965 - Kahane and Katznelson : for every null set N, there exists a continuous function such that its wavelet series diverges on every $x \in N$.

Une fonction continue et 2π-périodique dont la série de Fourier diverge en 0

Gourdon, *Analyse*, page 262

Exercice :

1. Soit $f : \mathbb{R} \to \mathbb{R}$ la fonction paire, 2π-périodique, telle que

$$\forall x \in [0,\pi],\ f(x) = \sum_{p=1}^{+\infty} \frac{1}{p^2} \sin\left[\left(2^{p^3}+1\right)\frac{x}{2}\right]$$

   Vérifier l'existence et la continuité de f sur $\mathbb{R}$.

2. Pour tout $\nu \in \mathbb{N}$, on pose

$$\forall n \in \mathbb{N},\quad a_{n,\nu} = \int_0^{\pi} \cos nt \sin\frac{(2\nu+1)t}{2}\,dt, \qquad \forall q \in \mathbb{N},\ s_{q,\nu} = \sum_{i=0}^{q} a_{i,\nu}$$

   Calculer explicitement les $a_{n,\nu}$, montrer que $s_{q,\nu} \geq 0$ pour tout (q,ν), et montrer l'existence d'une constante $B > 0$ telle que $s_{\nu,\nu} > B \ln \nu$ pour tout $\nu \in \mathbb{N}^*$.

3. Montrer que la série de Fourier de f diverge en 0.

Une série de Fourier-Lebesgue divergente presque partout.

Par

A. Kolmogoroff (Moscou).

Le but de cette Note est de donner *un exemple d'une fonction sommable* [1]) *dont la série de Fourier diverge presque partout* (c'est-à-dire: partout sauf aux points d'un ensemble de mesure nulle).

La fonction construite dans cette note est à carré non sommable et je ne sais rien sur l'ordre de grandeur des coefficients de sa série de Fourier. Les méthodes employées ici ne permettent pas de construire une série de Fourier divergente partout.

I. Je vais démontrer plus loin l'existence d'une suite de fonctions: $\varphi_1(x), \varphi_2(x) \ldots \varphi_n(x) \ldots$ définies pour $0 \leqq x \leqq 2\pi$ et jouissant de propiétés suivantes:

1873 - Du Bois-Reymond : The Fourier series of a continuous funciton may diverge at some points.

Theorem

1909 - Haar : This property is not inherent to any decomposition in an orthogonal basis.

Haar basis :

$$\begin{cases} \varphi \\ 2^{j/2}\psi(2^j \cdot -k),\ j \geq 0, k = 0...2^j - 1 \end{cases}$$

where $\varphi = \mathbf{1}_{[0,1]}$ and $\psi = \mathbf{1}_{[0,1/2]} - \mathbf{1}_{[1/2,1]}$

Proof. It is easy to see that the system in orthonormal. Consider for $j \geq 0$

$$V_j = \{2^{j/2}\varphi(2^j \cdot -k),\ k = 0..2^j - 1\} = \{2^{j/2}\mathbf{1}_{[\frac{k}{2^j},\frac{(k+1)}{2^j}]}, k = 0..2^{j-1}\}$$

and remark that

$$V_{j+1} = V_j \overset{\perp}{\oplus} W_j$$

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3. $\|f - P_{V_j} f\|_\infty \to 0$ when $j \to +\infty$
4. In particular, the Haar series of a continuous function f converges uniformly to f.

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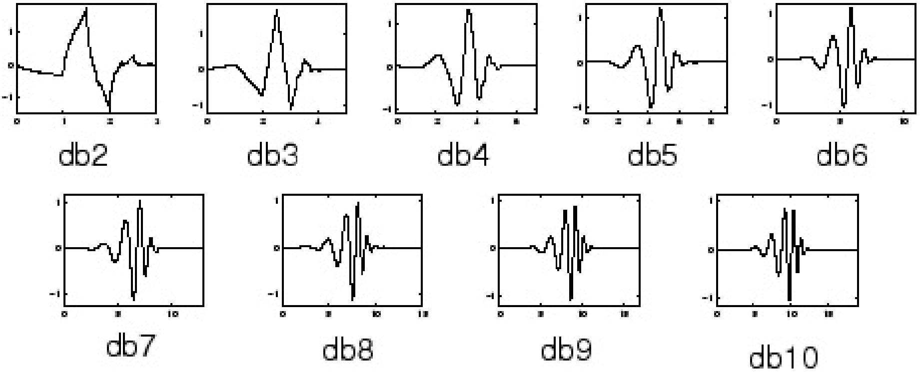
1980's : Wavelet Analysis, Morlet, Meyer, Daubechies, Mallat

We can construct **multiresolution analysis** (sets V_j and W_j) generated by functions φ and ψ such that

1. The system

$$\{\varphi(\cdot - k),\, k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j \cdot -k),\, j \in \mathbb{N},\, k \in \mathbb{Z}\}$$

   is an orthonormal basis of $L^2(\mathbb{R})$
2. The functions φ and ψ can be as regular as we want, with compact support and ψ has with null moments ($\int x^m \psi(x)dx = 0,\, 0 \leq m \leq M$)
3. The functions φ and ψ can be C^∞ with fast decay and ψ has all its moments null.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Definition

A sequence (e_n) of a separable Banach space X is a **Schauder basis** if for any $f \in X$ there exists a unique sequence (a_n) such that

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The sequence (e_n) is an **unconditional basis** of X if convergence also takes place after permutation of the elements of the series, or equivalently

- the series $\sum_n \epsilon_n a_n e_n$ converges in X for any choice of signs $\epsilon_n = \pm 1$.
- there exists $C > 0$ such that for any finite subset $N \subset \mathbb{N}$, any real numbers (a_n) and for any choice of signs (ϵ_n)

$$\| \sum_{n \in N} \epsilon_n a_n e_n \|_X \leq C \| \sum_{n \in N} a_n e_n \|_X$$

- The series $\sum \lambda_n a_n e_n$ converges for any $(\lambda_n) \in \ell^\infty(\mathbb{N})$

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If $(a_n) \in \ell^2(\mathbb{N})$, then the series

$$\sum_n b_n e_n$$

is continuous for some (b_n) with $|b_n| > |a_n|$.

Consequence : the shrinkage of the Fourier coefficients can introduce discontinuities

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The result was extended by Nazarov to any orthonormal basis (e_n) such that $\|e_n\|_1 \geq C$.

Theorem (Paley and Zygmund - 1932)

For all $1 \leq p < +\infty$, the Rademacher Fourier series

$$\sum_n \epsilon_n \, a_n \cos(2\pi n t)$$

belongs almost surely to $L^p(\mathbb{T})$ iff $(a_n) \in \ell^2(\mathbb{N})$

$\rightarrow$ The randomization of a Fourier series has a regularization effect.

Wavelet bases (if regular enough) are unconditional bases of numerous spaces such as

1. Lebesgue spaces $L^p(\mathbb{R})$, for $1 < p < +\infty$
2. Sobolev spaces $H_p^s(\mathbb{R})$ for $s \in \mathbb{R},\ 1 < p < +\infty$
3. Besov spaces $B_{p,q}^s(\mathbb{R})$ for $s \in \mathbb{R}, p, q \in \mathbb{R}$

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Note that

- $L^\infty(\mathbb{R})$ is not separable
- $C(\mathbb{T})$ has a Schauder basis but does not have unconditional bases
- $L^1(\mathbb{R})$ has no unconditional basis

1. What results if we replace Rademacher sequences with gaussian variables ?
2. What results for L^∞ and or C^0 ?
3. What results on pointwise regularity ?

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Theorem (Maurey, Pisier - 1973)

Let X a Banach space that does not contain ℓ_n^∞ uniformly and $(\chi_n)_n$ be a sequence of random variables such that $\sup_n \mathbb{E}\,|\chi_n|^p < +\infty$ for all $0 < p < +\infty$. Then

$$\sum x_n \text{ converges unconditionaly in } X \Rightarrow \sum \chi_n x_n \text{ converges a.s. unconditionaly in} X.$$

$\Rightarrow L^p(\mathbb{T}), 1 < p < +\infty$ is stable for gaussian randomization of the coefficients of the wavelet series.

Theorem (Pisier and Marcus - 1981)

Consider the two random Fourier series

$$\sum_n \epsilon_n a_n e_n \quad \textit{et} \quad \sum_n X_n a_n e_n$$

where

- (ϵ_n) *is a sequence of I.I.D. Rademacher random variables*
- (X_n) *is a sequence of I.I.D. gaussian random variables*

Then, almost surely, both of them are continuous or both of them are unbounded.

Consider $\psi_{j,k} = \psi(2^j \cdot -k)$ and $c_{j,k}(f) = 2^j \int f(x)\psi_{j,k}(x)dx$

Proposition (C. Esser, S. Jaffard and B.V.)

There exists a function f continuous such that

$$f = \sum_{j,k} c_{j,k}(f)\psi_{j,k}$$

- *its wavelet series is normally convergent to f*
- *the series*

$$\sum_{j,k} \chi_{j,k} c_{j,k}(f)\psi_{j,k}$$

  is a.s. nowhere locally bounded, where $\chi_{j,k}$ are I.I.D. unbounded random variables

Consider $\psi_{j,k} = \psi(2^j \cdot - k)$ and $c_{j,k}(f) = 2^j \int f(x)\psi_{j,k}(x)dx$

Proposition (C. Esser, S. Jaffard and B.V.)

There exists a function f continuous such that

$$f = \sum_{j,k} c_{j,k}(f)\psi_{j,k}$$

- *its wavelet series is normally convergent to f*
- *the series*

$$\sum_{j,k} \chi_{j,k} c_{j,k}(f)\psi_{j,k}$$

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If the wavelet ψ is continuous,
• the Rademacher randomization of f is continuous,
• the gaussian randomization of f is nowhere locally bounded

Proposition

Let f be in $L^\infty(\mathbb{T})$, if the wavelet is 0-smooth then the sequence ω_j defined by

$$\omega_j = \sup_k |c_{j,k}|$$

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Conversely, given a non-negative sequence (ω_j)
- *if $(\omega)_j \in \ell^1$ and $\omega_j = \sup_k |c_{j,k}|$ then the wavelet series converges to a continuous function*
- *if $(\omega_j) \notin \ell^1$, there exists $(c_{j,k})$ such that f is nowhere locally bounded.*

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Proof. (with compactly supported wavelets functions). It is easy to see that $f_j = \sum_k c_{j,k}\psi_{j,k}$ is bounded by $C\omega_j$, hence converges normally.

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Lemma

If (ω_j) is a non-negative sequence such that $\sum_j \omega_j = +\infty$ then there exists a subsequence j_n, such that $j_{n+1} - j_n \to +\infty$ and $\sum_{j_n} \omega_{j_n} = +\infty$.

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Proof. Immediate if we only require $j_{n+1} - j_n = N$, for a integer N.
One builds the sequence in the following way : we start with (j_n) obtained for $N = 2$ until the sum is greater than 1, then with elements for $N = 3$ such that the corresponding sum is greater than 1 and so on...

Let (ω_j) be in ℓ^∞ but not in ℓ^1 and let $\psi_{j,k}$ be compactly supported. Let (j_n) be given by the lemma : $\sum_{j_n} \omega_{j_n} = +\infty$
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$$c_{j,k} = \begin{cases} \omega_j & \text{for only one selected index k} \\ 0 \text{ otherwise} \end{cases}$$

- k is selected in the following way : for n great enough, the support of $\psi_{j_{n+1},k}$ is included in $\frac{1}{2}K_{j_n,k_n}$
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and for $j \notin (j_n)$, k is chosen such that $Supp\psi_{j,k} \cap K = \emptyset$
- It gives a function with one discontinuity
- One can duplicate the discontinuity by translation.

Proof of the theorem (1/2) : Existence of a continuous function whose randomization is a.s. nowhere bounded.

- Consider χ of same law of $\chi_{j,k}$
- For all n, one has $\mathbb{P}(|\chi| \geq n^3) > 0$.

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- Borel-Cantelli lemma gives a.s. the existence of an infinite number of (j_n, k_n) such that $\mathbb{P}(|\chi_{j,k}| \geq n^3)$

Proof of the theorem (2/2)

- Consider the series

$$f = \sum_{j_n} \frac{1}{n^2} \sum_{k} \psi_{j_n,k}$$

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- Since the argument is also true for any dyadic cube $\lambda \subset \mathbb{T}$, a.s., X_f is nowhere locally bounded.

Theorem

Let $(\chi_{j,k})$ be a sequence of I.I.D. unbounded random variables. For "almost every function" in $\mathcal{C}(\mathbb{T})$, the associated randomized wavelet series is almost surely nowhere locally bounded.

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- It provides an extension of the notion of "Lebesgue almost everywhere" in infinite dimensional spaces
- In infinite dimensional normed spaces, no non-zero Borel measure is both σ-finite and translation invariant
- In $\mathbb{R}^n$ a Borel set B has Lebesgue-measure 0 iff there exists a compactly supported probability measure μ such that $\mu(B+x)=0$ for all $x \in X$

Let X be a Banach space.

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• If $\mathcal{P}$ is a property that can be satisfied by points of X, one can prove that $\mathcal{P}$ only holds for a Haar-null set by exhibiting a stochastic process whose sample paths live in a compact subset of X and such that

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We take for Y the Rademacher randomization of the function which had unbounded gaussianization.

Assume that ψ is continuous. If the sequence $(\omega_j)_{j\in\mathbb{N}}$ satisfies

$$\sum_j \sqrt{j}\omega_j < +\infty$$

then the gaussian randomization of a wavelet series for which $\omega_j = \sup_j |c_{j,k}|$ is a.s. continuous.

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Proposition

Assume that ψ is compactly supported. There exists a wavelet series such that

$$\sum_{j\geq 3} \frac{\sqrt{j}}{\log(\log(j))} < +\infty$$

and its gaussian randomization is a.s. nowhere locally bounded.

An example between the difference of gaussian randomization of a Fourier series and of a wavelet series.
Let $\{x\}$ be the ”sawtooth function”

$$\{x\} = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Its Fourier series is

$$\{x\} = -\sum_{m=1}^{+\infty} \frac{sin(2\pi m x)}{\pi m}.$$

Wiener expansion of the Brownian motion on $\mathbb{T}$:

$$B(x) = \sqrt{2}\chi_0 x + \sum_{m=1}^{+\infty} \chi_m \frac{sin(2\pi m x)}{\pi m},$$

where (χ_m) is a sequence of I.I.D. standard gaussian random variables.

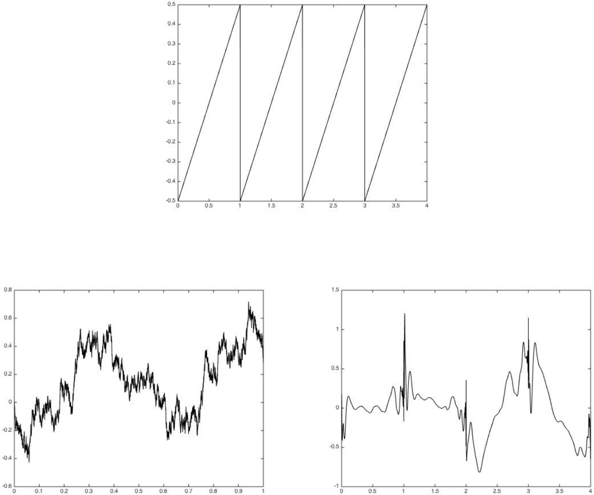

Figure – Effect of the randomization of the Sawtooth function

Definition

Let $x_0 \in \mathbb{R}$ and $h > 0$. A locally bounded function $f : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{C}^h(x_0)$ if there exists $C > 0$ and a polynomial P_{x_0} with $\deg P_{x_0} < \lfloor h \rfloor$ such that

$$|f(x) - P_{x_0}(x)| \leq C|x - x_0|^h$$

on a neighborhood of x_0. The *pointwise Hölder exponent* of f at x_0 is

$$h_f(x_0) = \sup\{h \geq 0 : \ f \in \mathcal{C}^h(x_0)\}.$$

The *iso-Hölder sets* of f are defined for every $h \in [0, +\infty]$ by

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Definition

The **multifractal spectrum** $\mathcal{D}_f$ of a locally bounded function f is the function

$$\mathcal{D}_f : h \in [0, +\infty] \mapsto \dim_{\mathcal{H}} I_f(h)$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

Definition

Let λ be a dyadic interval and 3λ the interval of same center as λ and 3 times wider. If f is a bounded function, the wavelet leader d_λ of f is defined by

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|.$$

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Theorem (2004 - Jaffard)

Let $h > 0$ and $x_0 \in \mathbb{R}$. Assume that f is a bounded function and that the wavelet has r vanishing moments with $r > \lfloor h \rfloor + 1$.

- *If f belongs to $\mathcal{C}^h(x_0)$, then there exists $C > 0$ such that*

$$\forall j \geq 0, \quad d_{\lambda_j(x_0)} \leq C 2^{-hj}. \tag{1}$$

- *Conversely, if (1) holds and if f is uniformly Hölder (i.e. f belongs to $\mathcal{C}^\varepsilon(\mathbb{R})$ for some $\varepsilon > 0$), then f belongs to $C^{h'}(x_0)$ for all $h' < h$.*

In particular, if $f \in \mathcal{C}^\varepsilon(\mathbb{R})$ for some $\varepsilon > 0$, then

$$h_f(x_0) = \liminf_{j \to +\infty} \frac{\log d_{\lambda_j(x_0)}}{\log 2^{-j}}.$$

Let us now consider two parameters $\alpha > 0$ and $\eta \in (0, 1)$.

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The signal $f = \sum_{j,k} 2^{-j\alpha}\psi_{j,k}$ is monofractal :

$$\mathcal{D}_f(h) = \begin{cases} 1 & \text{if } h = \alpha \\ 0 & \text{otherwise} \end{cases}$$

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The Lacunary Wavelet Series (denoted by LWS) on $\mathbb{T}$ of parameters α and η is the process defined by

$$F_{\alpha,\eta} = \sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k} \quad \text{with} \quad c_{j,k} = 2^{-\alpha j}\xi_{j,k}$$

where $(\xi_{j,k})_{j,k}$ denotes a sequence of independent random Bernoulli variables of parameter $2^{(\eta-1)j}$.

Theorem

(S. Jaffard) If $\eta < 1$, $F_{\alpha,\eta}$ is multifractal and almost surely, one has

$$\mathcal{D}_{F_{\alpha,\eta}}(h) = \rho_{F_{\alpha,\eta}}(h) = \begin{cases} \frac{\eta}{\alpha} h & \textit{if } h \in \left[\alpha, \frac{\alpha}{\eta}\right], \\ -\infty & \textit{otherwise.} \end{cases}$$

Proof. *Step 1* : Almost surely, there is $J \in \mathbb{N}$ such that

$$d_\lambda \geq \sup_{\lambda' \subset \lambda} |c_{\lambda'}| \geq 2^{-\frac{\alpha}{\eta}(j+\log_2 j)}$$

for every λ at a scale $j \geq J$. In particular, $h_f(x) \leq \frac{\alpha}{\eta}$ for every $x \in [0, 1]$.

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Step 2 : For every $\delta \in (0,1]$, consider the random sets

$$E_\delta = \limsup_{j \to +\infty} \bigcup_{k \,:\, c_{j,k} \neq 0} B\left(k2^{-j}, 2^{-\delta j}\right),$$

- If $x \in E_\delta$, then $h_{F^d_{\alpha,\eta}}(x) \leq \frac{\alpha}{\delta}$.
- If $h_{F^d_{\alpha,\eta}}(x) < \frac{\alpha}{\delta}$, then $x \in E_\delta$.

We need to compute the Hausdorff dimension of $\limsup$ of union of balls.

Theorem (General mass transference principle, Beresnevich,Velani)

Let X be a compact set in $\mathbb{R}^n$ and assume that there exist $s \leq n$ and $a, b, r_0 > 0$ such that

$$ar^s \leq \mathcal{H}^s(B \cap X) \leq br^s \tag{2}$$

for any ball B of center $x \in X$ and of radius $r \leq r_0$. Let $s' > 0$. Given a ball $B = B(x, r)$ with center in X, we set

$$B^{s'} = B\left(x, r^{\frac{s'}{s}}\right).$$

Assume that $(B_n)_{n \in \mathbb{N}}$ is a sequence of balls with center in X and radius r_n such that the sequence $(r_n)_{n \in \mathbb{N}}$ converges to 0. If

$$\mathcal{H}^s\left(X \cap \limsup_{n \to +\infty} B_n^{s'}\right) = \mathcal{H}^s(X),$$

then

$$\mathcal{H}^{s'}\left(X \cap \limsup_{n \to +\infty} B_n\right) = \mathcal{H}^{s'}(X).$$

Definition

Let $E \subset \mathbb{R}$ and $\delta > 0$. For $s \in [0,1]$, set

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i\in\mathbb{N}} \mathrm{diam}(A_i)^s : E \subset \bigcup_{i\in\mathbb{N}} A_i \text{ and } \mathrm{diam}(A_i) < \delta \ \forall i \in \mathbb{N}\right\}.$$

The δ-dimensional Hausdorff measure of E is $\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E)$ and the Hausdorff dimension of E is given by

$$\dim_{\mathcal{H}}(E) = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = +\infty\}.$$

We use the usual convention that $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

Proposition

Let $f = \sum C_{j,k}\psi_{j,k}$ be a uniform Hölder wavelet series, and let

$$X_f = \sum C_{j,k}\chi_{j,k}\psi_{j,k}$$

bet its Gaussian randomization. Then

$$\text{a. s. } \forall t \qquad h_f(t) \leq h_{X_f}(t).$$

Consider X_1 a Lacunary wavelet series and X_2 the gaussianization of a sample path $X_1(\omega)$:

$$X_2 = \sum_{j,k} 2^{-j\alpha} \chi_{j,k} \psi_{j,k}$$

for (j, k) such that $c_{j,k}(X_1(\omega)) \neq 0$.

Theorem (C. Esser, S. Jaffard, B.V)

One has

$$\mathcal{D}_{X_1} = \mathcal{D}_{X_2}$$

but

$$\{t,\ h_{X_1}(t) \neq h_{X_2}(t)\}$$

is of full measure.

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2. Is it possible to construct a process for which the gaussianized process has a random spectrum ?
3. What about the bivariate spectrum

$$\mathcal{D}(h_1, h_2) := dim_H\{t, h_{X_1}(t) = h_1 \text{ and } h_{x_2}(t) = h_2\}?$$