Data-Driven Methods in Control: Error Bounds and Guaranteed Stability

Manuel Schaller

Workshop and Summer School on Applied Analysis 2025

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UNIVERSITY OF TECHNOLOGY
IN THE EUROPEAN CAPITAL OF CULTURE
CHEMNITZ

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German Research Foundation

A dynamical system

Consider the **discrete and scalar model**

$$x_{k+1} = 2x_k, \qquad k \in \mathbb{N},$$
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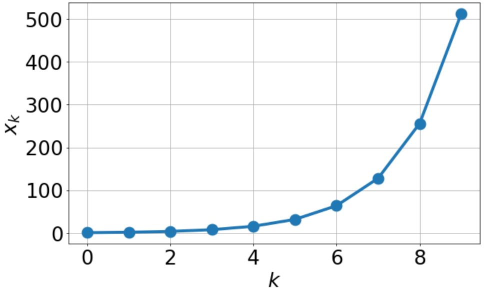

An attempt to control

Consider now a **controlled model**

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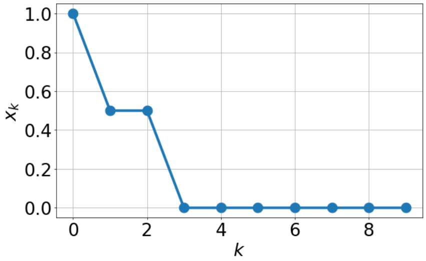

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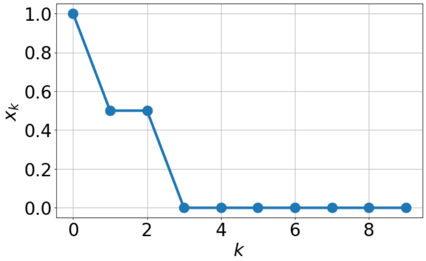

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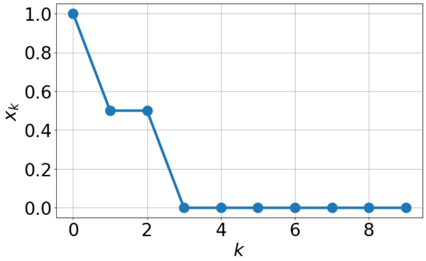

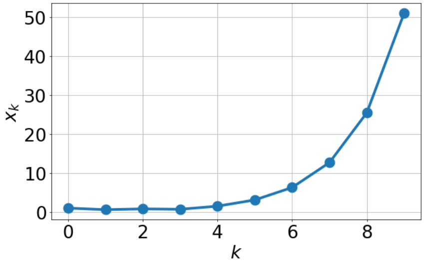

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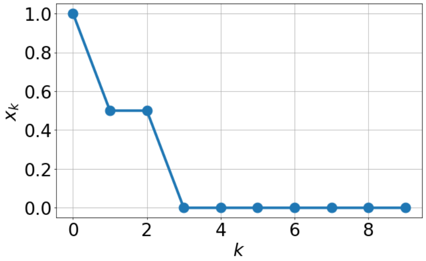

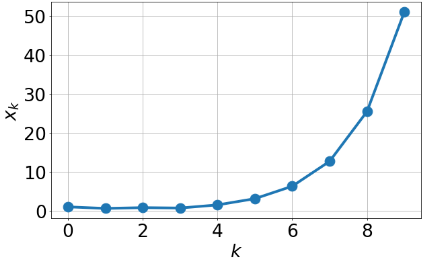

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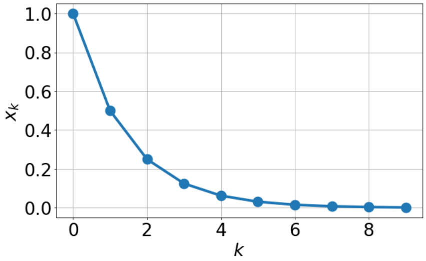

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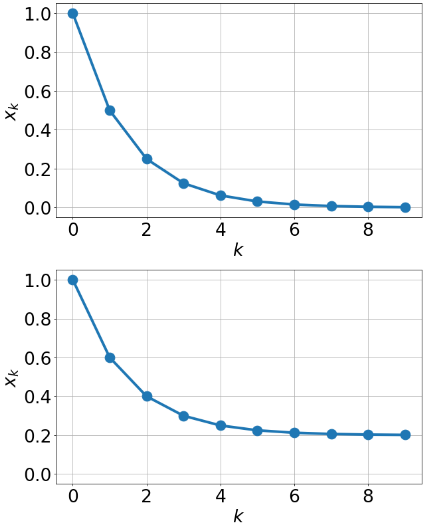

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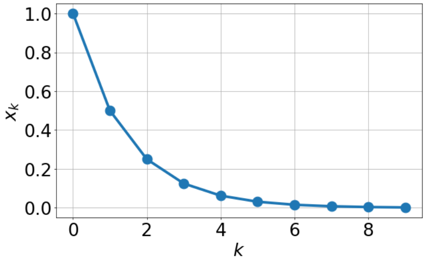
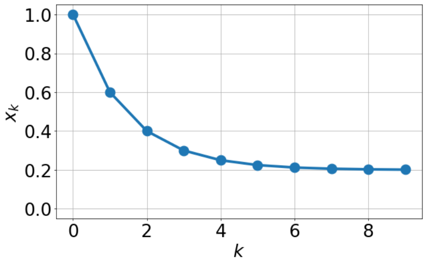
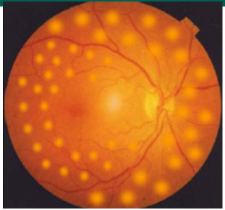




Retinal Photocoagulation

Laser treatment for retinal diseases (e.g. **macular edema**)



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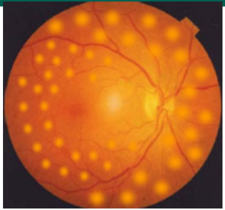

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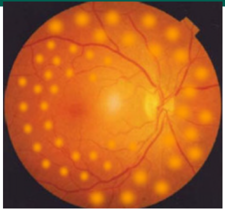

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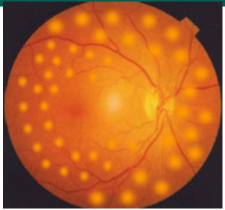

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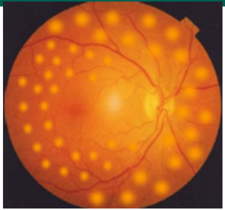

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$\hookrightarrow$ **Optimal control** to guarantee effective and safe treatment

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Model-based planning via optimal control

$$\partial_t x(t,\omega) = \Delta x(t,\omega) + B(\omega,p)u(t)$$
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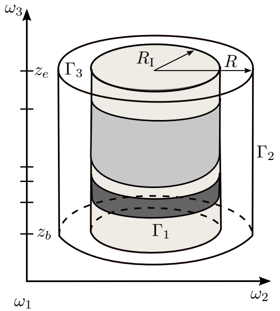

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$$x(t,\omega) \le 55°$$
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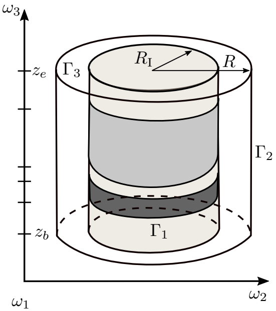

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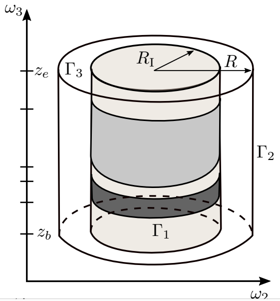

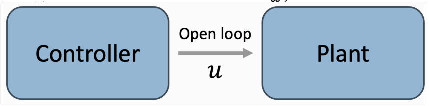

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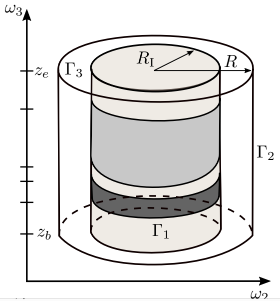

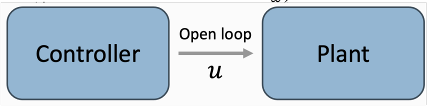

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- Patient behavior, e.g., **saccades**.
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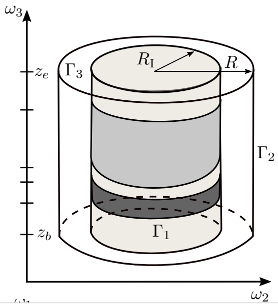

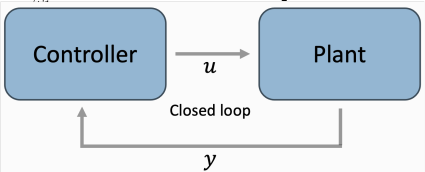

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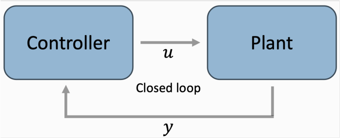
Need for feedback control.

Experiments with pig eyes



Feedback loop in **10 kHz**:

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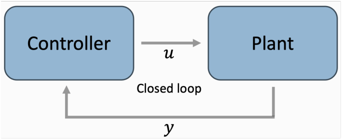

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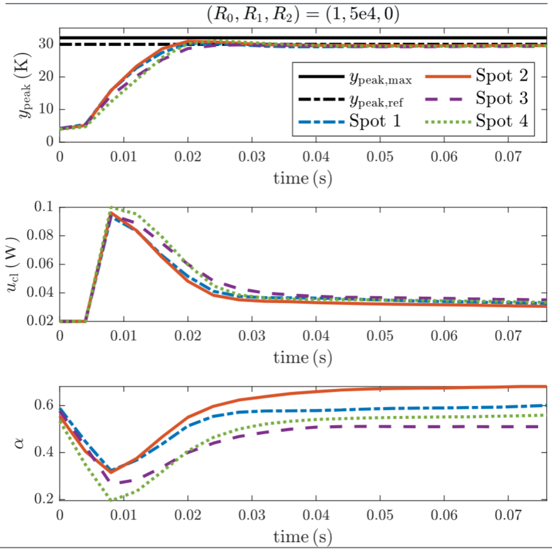

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Controller design for $\widehat{F}$: What can we say about F?

Outline

1. Today: Stability guarantees via kernel methods
2. Friday: Koopman operator-based techniques

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Given ODE

$$\dot{x} = f(x), \qquad x(0) = x^0$$

with associated flow $\varphi(t; x^0)$. Then, for fixed $\Delta t > 0$, we may define

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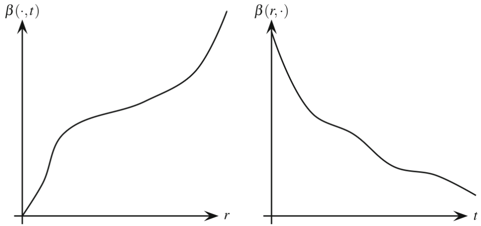
Tools for stability analysis

Definition (Comparison functions)

- $\mathcal{K} := \{\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \,|\, \alpha$ continuous, strictly increasing and $\alpha(0) = 0\}$.
- $\mathcal{K}_\infty := \{\alpha \in \mathcal{K} \,|\, \alpha$ unbounded$\}$.
- $\mathcal{KL} := \{\beta : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ cont. $|\, \forall t \geq 0\ \beta(\cdot, t) \in \mathcal{K}_\infty$ and $\forall r > 0 : \beta(r, \cdot)$ strictly decreasing and $\lim_{t\to\infty} \beta(r,t) = 0\}$.



Grüne, Pannek, 2017

Stability notions

Definition

Equilibrium $x^* = F(x^*)$ asymptotically stable with domain of attraction $Y \subset \mathbb{R}^n$ if $\exists \beta \in \mathcal{KL}$:

$$\forall x \in Y, n \in \mathbb{N}_0 : \quad \|F^n(x) - x^*\| \leq \beta(\|x - x^*\|, n). \tag{1}$$

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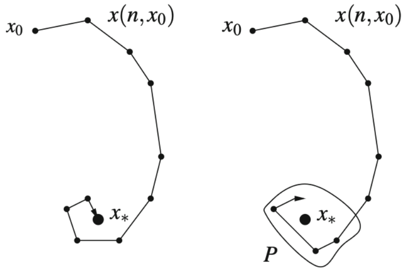

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Lyapunov functions

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A continuous function $V : Y \subset \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lyapunov function if $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_V \in \mathcal{K}$:

$$\alpha_1(\|x - x^*\|) \leq V(x) \leq \alpha_2(\|x - x^*\|) \qquad \forall\, x \in Y$$

and

$$V(F(x)) \leq V(x) - \alpha_V(\|x - x^*\|) \qquad \forall x \in Y. \tag{2}$$

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Proposition

Let Y forward invariant, $Y \ni x^* = F(x^*)$.

- *If V Lyapunov function then x^* asymptotically stable.*
- *If $P \ni x^*$ forward invariant s.t. decrease (2) holds on $S = Y \setminus P$, then P-practically as. stable.*

Sketch of the proof

Quadratic setting: Assume that $\alpha_V(r) = c_V r^2, \alpha_1(r) = c_1 r^2, \alpha_2(r) = c_2 r^2$ for $c_V, c_1, c_2 > 0$.

Sketch of the proof

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Remark A similar but more technical argument also works for $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_V \in \mathcal{K}$.

Asymptotic stability

Assume in the following that V has modulus of continuity s.t. $|V(x) - V(y)| \leq \omega_V(\|x - y\|)$.

Bold, Philipp, S., Worthmann, to appear in SIAM Journal on Control and Optimization, 2025

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Assume there is a Lyapunov function for $\widehat{F}$ w.r.t. $x^* \in \mathbb{R}^n$ and

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In the quadratic case $\alpha_V(r) = c_V r^2$ and if V has Lipschitz constant L_V, then $\eta = \frac{2L_V}{c_V}$

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A wish: proportional error

$$\cancel{\|F(x) - \widehat{F}(x)\| \leq \varepsilon} \quad \rightsquigarrow \quad \|F(x) - \widehat{F}(x)\| \leq \varepsilon \|x - x^*\|$$

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If decrease at least as strong as modulus of continuity: Asymptotic stability.

Asymptotic stability

Corollary

Let $\widehat{F}$ be asymptotically stable with Lyapunov function V such that

$$\limsup_{r \searrow 0} \frac{\omega_V(r)}{\alpha_V(r)} < \infty.$$

and

$$\|F(x) - \widehat{F}(x)\| \leq \varepsilon \|x - x^*\|.$$

Then F is also asymptotically stable with Lyapunov function V.

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Norm-based Lyapunov functions

Assume $V(x) = x^2$ and $\alpha_V(r) = cr^2$ for some $c > 0$.

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the compatibility assumption does not hold.

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Remark

If $V(x) = \|x - x^*\|^p$ *for some* $p \in \mathbb{N}$*, then*

$$\limsup_{r \searrow 0} \frac{r^p}{\alpha_V(r)} < \infty$$

is sufficient.

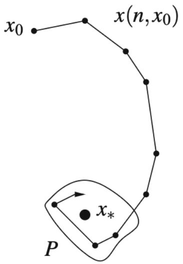
Intermediate summary

Practical asymptotic stability

$$\|F(x) - \hat{F}(x)\| \leq \varepsilon$$

allow to infer

$$\hat{F} \text{ as. stab.} \Rightarrow F \text{ prac. as. stab.}$$



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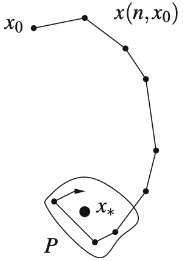

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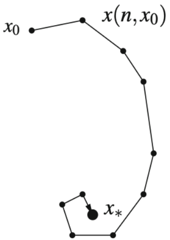

Kernel-based approximations

Reproducing Kernel Hilbert Spaces (RKHS)

A **RKHS $\mathbb{H}$ over $\Omega \subset \mathbb{R}^n$** is a

- Hilbert space of functions $f : \Omega \to \mathbb{R}$
- with s.p.d. kernel $k : \Omega \times \Omega \to \mathbb{R}$ with $k(x, \cdot) \in \mathbb{H}$ for all $x \in \Omega$ and

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- Thin-Plate splines (Beppo Levi spaces)

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Data-driven approximations

Given data points $\mathcal{X} = \{x_1 \ldots, x_d\} \subset \Omega$ and set

$$V_{\mathcal{X}} := \operatorname{span}\{k(x_1, \cdot), k(x_2, \cdot), \ldots, k(x_d, \cdot)\} \subset \mathbb{H}$$

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Easy to compute: Basis representation $v(x) = \sum_{i=1}^d \alpha_i k(x_i, x)$ satisfies

$$\sum_{i=1}^d \alpha_i k(x_i, x_j) = f(x_j) \quad \rightsquigarrow \quad \alpha = \mathbf{K}_{\mathcal{X}}^{-1} f_{\mathcal{X}}$$

with

$$(f_{\mathcal{X}})_i = f(x_i), \quad (\mathbf{K}_{\mathcal{X}})_{ij} = k(x_i, x_j) \quad \text{s.p.d.}.$$

A first simple data-driven model

Given $(x_i, F(x_i))_{i=1}^d$ and define best approximation componentwise

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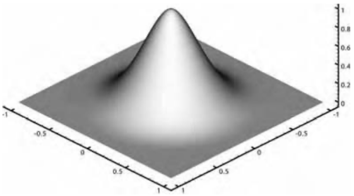
Projection error controlled by fill distance

$$h_\mathcal{X} := \sup_{x\in\Omega} \min_{1\leq i\leq d} \|x - x_i\|_2,$$

Wendland radial basis functions

Function	Smoothness
$\phi_{1,0}(r) = (1-r)_+$	C^0
$\phi_{1,1}(r) \doteq (1-r)_+^3(3r+1)$	C^2
$\phi_{1,2}(r) \doteq (1-r)_+^5(8r^2+5r+1)$	C^4
$\phi_{3,0}(r) = (1-r)_+^2$	C^0
$\phi_{3,1}(r) \doteq (1-r)_+^4(4r+1)$	C^2
$\phi_{3,2}(r) \doteq (1-r)_+^6(35r^2+18r+3)$	C^4
$\phi_{3,3}(r) \doteq (1-r)_+^8(32r^3+25r^2+8r+1)$	C^6



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Wendland, Advances in Computational Mathematics 1995

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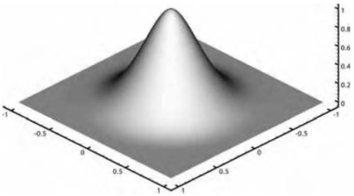

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and

$$\mathbb{H} \cong H^{\sigma_{n,k}}(\Omega).$$

Wendland, Advances in Computational Mathematics 1995

An error bound

Theorem (Wendland 1995)

There are $C, h_0 > 0$ *such that for every set* $\mathcal{X} = \{x_i\}_{i=1}^d \subset \Omega$ *with* $h_\mathcal{X} \leq h_0$ *and all* $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq k$,

$$|D^\alpha \varphi(x) - D^\alpha (P_\mathcal{X}\varphi)(x)| \leq C h_\mathcal{X}^{k+1/2-|\alpha|} \|\varphi\|_{\mathbb{H}_{\Phi_{n,k}}} \qquad \forall x \in \Omega$$

In particular, with $\alpha = 0$,

$$\|I - P_\mathcal{X}\|_{\mathbb{H}_{\Phi_{n,k}} \to C_b(\Omega)} \leq C h_\mathcal{X}^{k+1/2}.$$

An error bound

Theorem (Wendland 1995)

There are $C, h_0 > 0$ *such that for every set* $\mathcal{X} = \{x_i\}_{i=1}^d \subset \Omega$ *with* $h_\mathcal{X} \leq h_0$ *and all* $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq k$,

$$|D^\alpha \varphi(x) - D^\alpha (P_\mathcal{X} \varphi)(x)| \leq C h_\mathcal{X}^{k+1/2-|\alpha|} \|\varphi\|_{\mathbb{H}_{\Phi_{n,k}}} \qquad \forall x \in \Omega$$

In particular, with $\alpha = 0$,

$$\|I - P_\mathcal{X}\|_{\mathbb{H}_{\Phi_{n,k}} \to C_b(\Omega)} \leq C h_\mathcal{X}^{k+1/2}.$$

Direct consequence :

$$\|F(x) - \widehat{F}(x)\| \leq C h_\mathcal{X}^{k+1/2} \|F\|_{\mathbb{H}^n}$$

Constant error bound $\rightsquigarrow$ practical asymptotic stability.

A proportional error bound

Theorem (BPSW 2024)

There are $C, h_0 > 0$ *such that, for every set* $\mathcal{X} = \{x_j\}_{j=1}^d \subset \Omega$ *with* $h_\mathcal{X} \leq h_0$,

$$\left|\varphi(x) - (P_\mathcal{X}\varphi)(x)\right| \leq C h_\mathcal{X}^{k-1/2} \operatorname{dist}(x, \mathcal{X}) \|\varphi\|_{\mathbb{H}_{\Phi_{n,k}}} \qquad \forall x \in \Omega.$$

Bold, Philipp, S., Worthmann, to appear in SIAM Journal on Control and Optimization, 2025

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$$\left|\varphi(x) - (P_\mathcal{X}\varphi)(x)\right| \le C h_\mathcal{X}^{k-1/2}\,\mathrm{dist}(x,\mathcal{X})\|\varphi\|_{\mathbb{H}_{\Phi_{n,k}}} \qquad \forall x \in \Omega.$$

Sketch of the proof: Set $e = \varphi - P_\mathcal{X}\varphi$. Then for $x \in \Omega$ and $z \in \mathcal{X}$

$$e(x) = \underbrace{e(z)}_{=0} + \underbrace{\nabla e(z)}_{\le C h_\mathcal{X}^{k-1/2}\|\varphi\|}(x - z) + \dots$$

Bold, Philipp, S., Worthmann, to appear in SIAM Journal on Control and Optimization, 2025

Stability

Corollary

If $x^* \in \mathcal{X}$:

- x^* *is equilibrium of* $\widehat{F}$ *iff* x^* *equilibrium of* F*:*

$$\widehat{F}(x^*) = P_{\mathcal{X}} F(x^*) = F(x^*).$$

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- *Proportional bound*

$$\|F(x) - \widehat{F}(x)\| \leq Ch^{k-1/2}\, \mathrm{dist}(x, \mathcal{X}) \|F\| \leq \tilde{c}\|x - x^*\|$$

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- *Suitable compatbility assumptions on the Lyapunov function* $\Rightarrow$ *asymptotic stability is preserved.*

Bold, Philipp, S., Worthmann, to appear in SIAM Journal on Control and Optimization, 2025

An example[1]

$$x^+ = F(x) := \frac{1}{8}\begin{pmatrix} \|x\|^2 - 1 & -1 \\ 1 & \|x\|^2 - 1 \end{pmatrix} x \quad \rightsquigarrow V(x) = \|x\|^2,\ \alpha_V(r) = 7r^2/32.$$

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[1]Li, Hafstein, Kellett, CDC 2014

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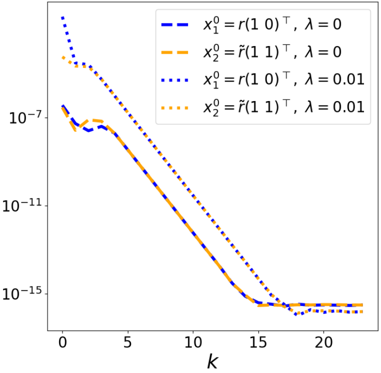

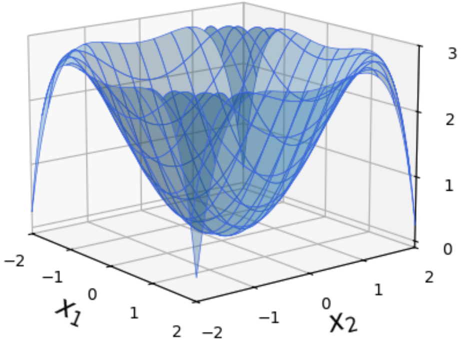

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Learning control systems

Control-affine dynamics

$$x^+ = F(x, u) = g_0(x) + G(x)u.$$

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Componentwise projection. Compute

$$H_{pq} \approx \widehat{H}_{pq} := \sum_{i=1}^{d} (\mathbf{K}_{\mathcal{X}}^{-1}(H_{pq})_{\mathcal{X}})_i k(x_i, x) \qquad \text{Best-approximation of } H_{pq}.$$

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Set $\begin{bmatrix} \widehat{g}_0 & \widehat{G} \end{bmatrix} = \widehat{H}$ and define

$$x^+ = \widehat{F}(x,u) = \widehat{g}_0(x) + \widehat{G}(x)u$$

Error bound

Corollary

There are $C, h_0 > 0$ *s.t. for every set* $\mathcal{X} = \{x_j\}_{j=1}^d \subset \Omega$ *with* $h_\mathcal{X} \leq h_0$*,*

$$\|F(x,u) - \widehat{F}(x,u)\|_\infty \leq C \cdot h_\mathcal{X}^{k-1/2} \operatorname{dist}(x,\mathcal{X}) \max_{p,q} \|H_{pq}\|_\mathbb{H} (1 + \|u\|_1) \qquad \forall x \in \Omega, u \in \mathbb{R}^m.$$

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Definition (Stabilizing feedbacks)

We say that a feedback law $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is asymptotically stabilizing a system $F(x,u)$ if the autonomous system

$$C(x) := F(x, \kappa(x))$$

is asymptotically stable towards x^*.

Feedback control

Example

$$x^+ = F(x,u) = 2x + u$$

Bold, Philipp, S., Worthmann, to appear in SIAM Journal on Control and Optimization, 2025

Feedback control

Example

$x^+ = F(x,u) = 2x + u$ Feedback $\kappa(x) = -1.5x$

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Feedback control

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$$x^+ = F(x,u) = 2x + u \text{ Feedback } \kappa(x) = -1.5x \Rightarrow \quad x^+ = F(x,\kappa(x)) = 2x - 1.5x = 0.5x.$$

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Corollary (Feedback control)

Given feedback law $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ bounded on bounded sets stabilizing $\widehat{F}$ with Lyapunov function satisfying compatibility assumption.

Then, if $x^* \in \mathcal{X}$, κ is also asymptotically stabilizing F.

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Proof: $\widehat{C}(x) = \widehat{F}(x,\kappa(x))$ is asymptotically stable.

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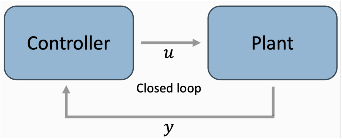
Data-driven Model Predictive Control

Experiments with pig eyes



Feedback loop in **10 kHz**:

1. **Solve** the optimal control problem with (x^0, p)
2. **Apply** optimal control u
3. Obtain **measurements** y
4. **State and parameter** estimation: Update (x^0, p).

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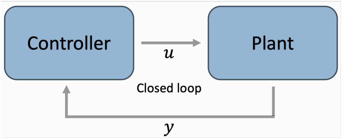

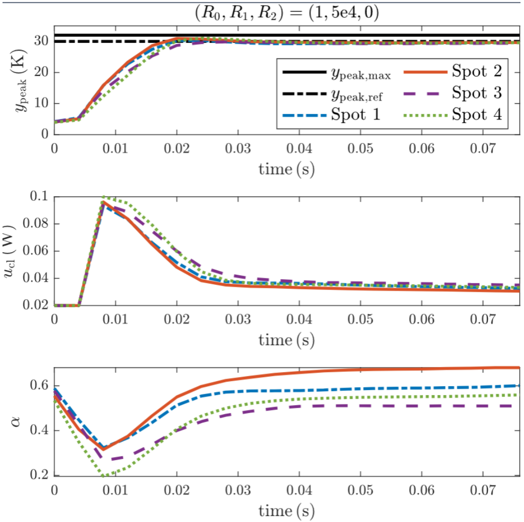

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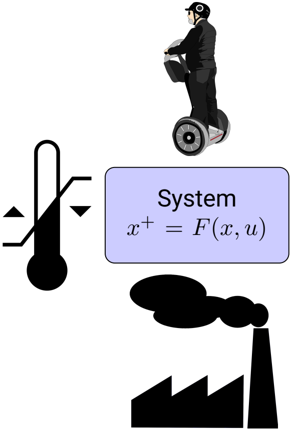
Model Predictive Control (MPC)



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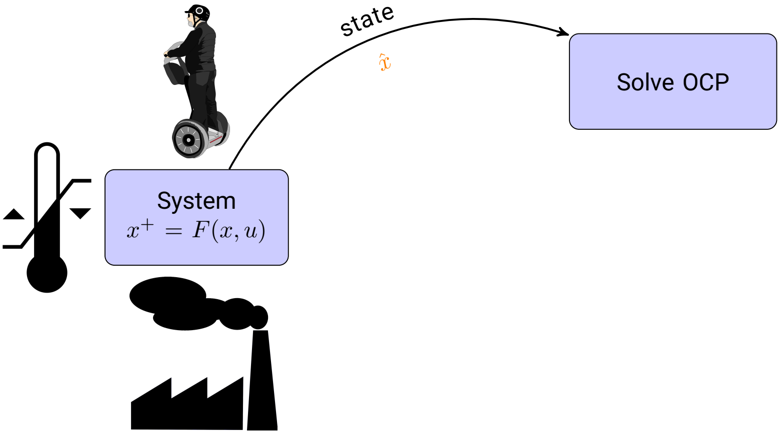

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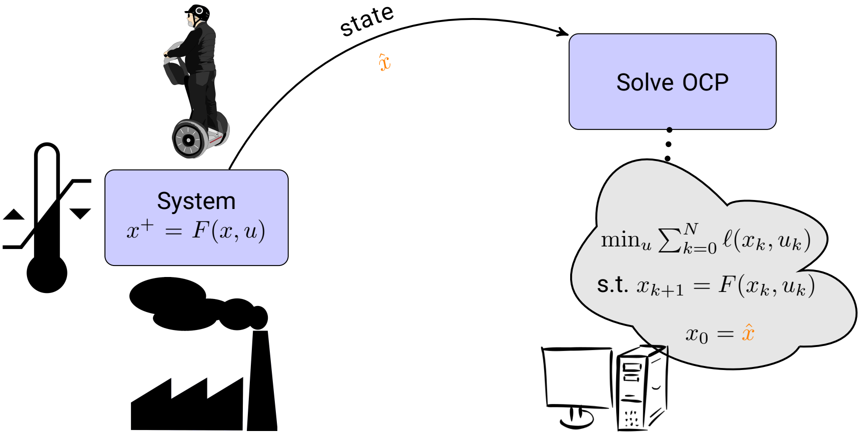

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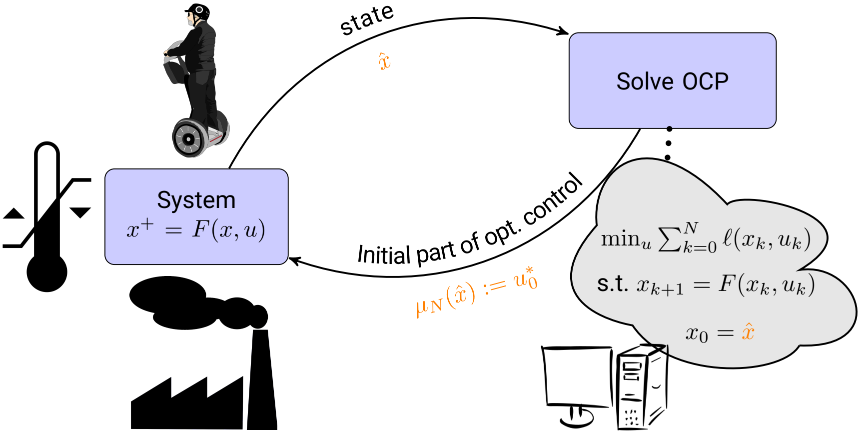

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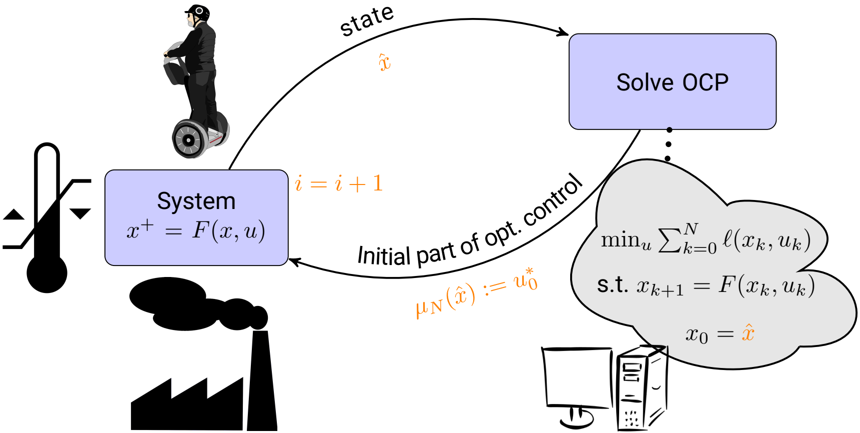

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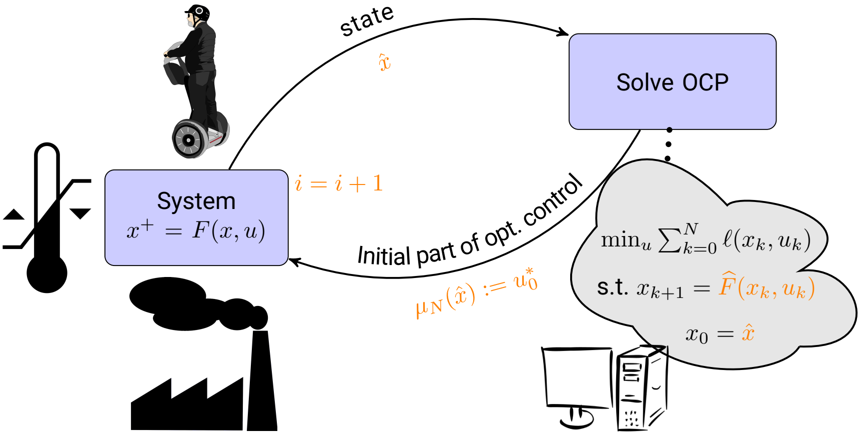

What happens, if the model in the OCP is a data-driven surrogate for the true dynamics?

Model Predictive Control

Set-point stabilization of the origin with $F(0,0) = 0$

MPC scheme with prediction horizon N

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Here: Optimization with data-driven surrogate $\widehat{F} \rightsquigarrow$ Closed loop: $F(\cdot, \widehat{\mu}_N(\cdot))$

How to prove stability?

Central tool: Optimal value function

$$V_N(x^0) = \inf_{u \in \mathcal{U}_N^\varepsilon(\hat{x})} \sum_{k=0}^{N-1} \|x_u(k;x^0)\|^2 + \|u(k)\|^2$$

Definition[2]

An OCP is cost controllable if

$$\exists \gamma > 0 : \quad V(x^0) \leq \gamma \cdot \|x^0\|^2 \qquad \forall N \geq 1, x^0 \in \mathbb{X}$$

[2] Coron, Grüne, Worthmann, SICON 2020

Relaxed Lyapunov inequality

Theorem

Let the OCP be cost controllable.

Schimperna, Worthmann, S., Bold, Magni, arXiv:2505.05951, 2025

Relaxed Lyapunov inequality

Theorem

Let the OCP be cost controllable. Then there is $\alpha \in (0,1]$ *and* $N \in \mathbb{N}$ *such that for all* $x \in S$*,*

$$V_N(F(x,\mu_N(x))) \leq V_N(x) - \alpha\ell(x,\mu_N(x))$$

In addition, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *s.t.* $\forall x \in S$*,* $u \in \mathbb{U}$*.*

$$\alpha_1(\|x\|) \leq V_N(x) \leq \alpha_2(\|x\|)$$

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$$\alpha_1(\|x\|) \leq V_N(x) \leq \alpha_2(\|x\|)$$

Corollary

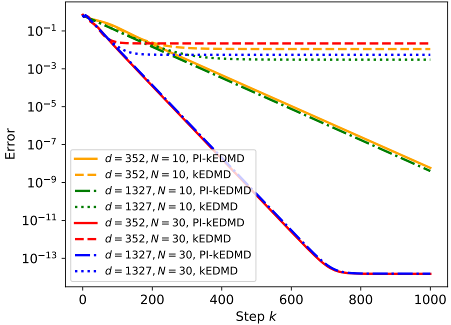
For small enough fill distance, the MPC controller achieves asymptotic stability.

Schimperna, Worthmann, S., Bold, Magni, arXiv:2505.05951, 2025

Numerical example

Van der Pol oscillator:

$$\dot{x} = \begin{pmatrix} x_2 \\ \nu(1-x_1)^2 x_2 - x_1 + u \end{pmatrix}$$



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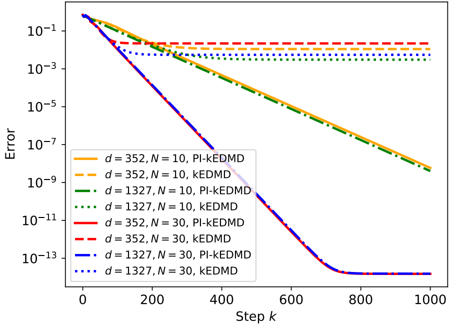

Four tank system:

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \\ \dot{h}_4 \end{pmatrix} = -\frac{\sqrt{2g}}{S} \begin{pmatrix} a_1\sqrt{h_1} + a_3\sqrt{h_3} \\ a_2\sqrt{h_2} + a_4\sqrt{h_4} \\ a_3\sqrt{h_3} \\ a_4\sqrt{h_4} \end{pmatrix} + \begin{bmatrix} \frac{\gamma_a}{S} & 0 \\ 0 & \frac{\gamma_b}{S} \\ 0 & \frac{1-\gamma_b}{S} \\ \frac{1-\gamma_a}{S} & 0 \end{bmatrix} \begin{pmatrix} q_a \\ q_b \end{pmatrix}.$$