Data-Driven Methods in Control: Error Bounds and Guaranteed Stability

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Koopman-based methods

Koopman-based predictions

For $\Omega \subset \mathbb{R}^n$ and $F : \Omega \to \Omega$, consider the dynamical system

$$x_{i+1} = F(x_i), \qquad x_0 \in \Omega.$$

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Predict observations from previous ones: $\varphi(x^+) = \mathcal{K}\varphi(x)$

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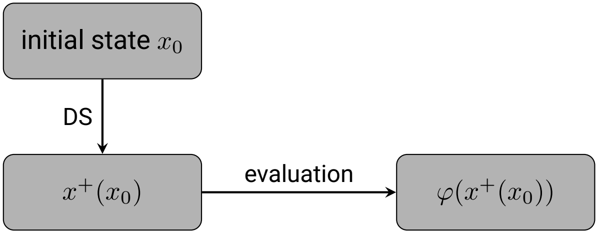
Illustration of the Koopman scheme



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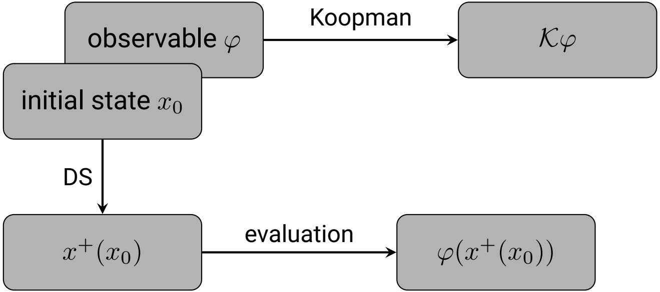

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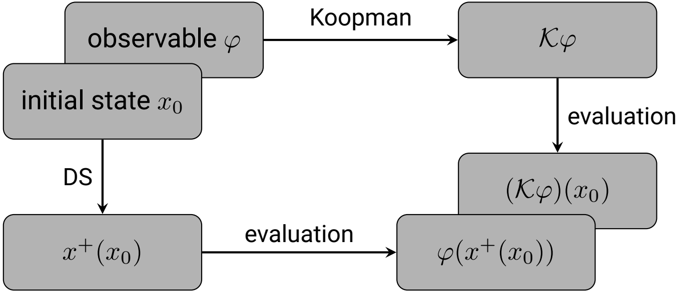

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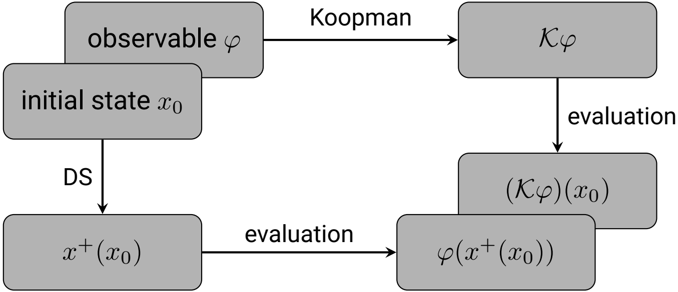

Old idea: Koopman 1931

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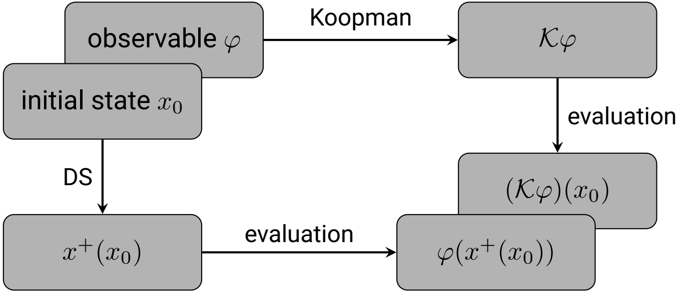

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Today: Powerful tools to approximate $\mathcal{K}$ based on data: Extended Dynamic Mode Decomposition

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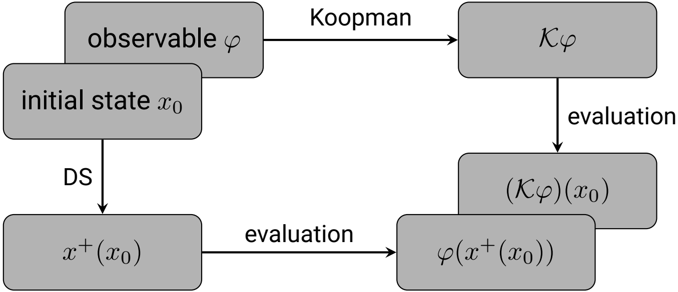


Old idea: Koopman 1931
Today: Powerful tools to approximate $\mathcal{K}$ based on data: Extended Dynamic Mode Decomposition

Successfully applied in fluid dynamics, epidemiology, neuroscience, video processing and molecular dynamics [Schmid '10], [Tu et al. '14], [Brunton, Kutz et al. '16] [Noé et al. '15]

Wake behind a cylinder

$$\dot{v} + (v \cdot \nabla)v = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2 v$$
$$\operatorname{div} v = 0$$

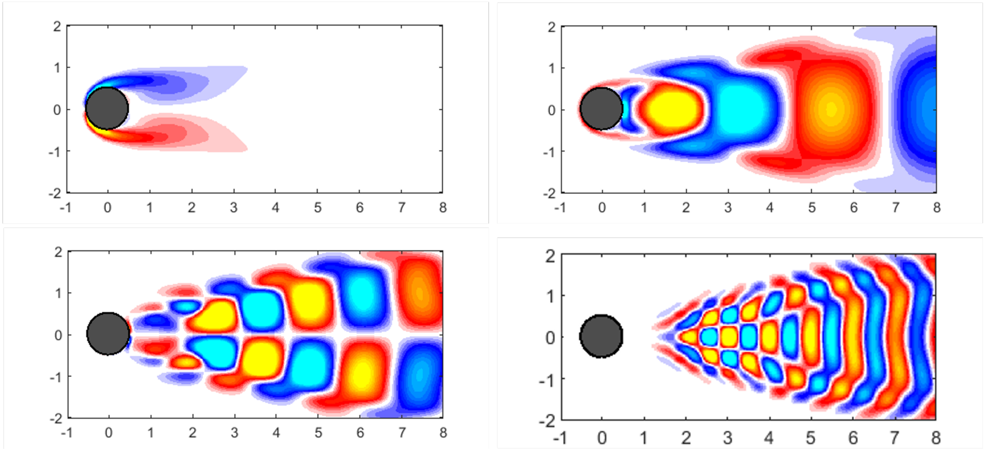
Click

Dynamic mode decomposition: data-driven modeling of complex systems, Kutz, Brunton, Brunton, Proctor, SIAM 2016
Thanks to Emilia Krendelsberger and Karl Worthmann (TU Ilmenau)

Koopman eigenfunctions

Let $\varphi : \Omega \to \mathbb{R}$ and $\lambda \in \mathbb{C}$ satisfy

$$\mathcal{K}\varphi = \lambda\varphi.$$



Model reduction

True

$r = 3$	$r = 11$	$r = 21$

Consequences of linearity

Eigenfunctions and Koopman mode decomposition: Let $\varphi : \Omega \to \mathbb{R}$ satisfy

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- Conserved quantities: $\lambda = 1$, i.e.,

$$\varphi(x_k) = (\mathcal{K}^k\varphi)(x) = \varphi(x_0) = c \qquad \forall x \in S_c = \{x : \varphi(x) = c\}.$$

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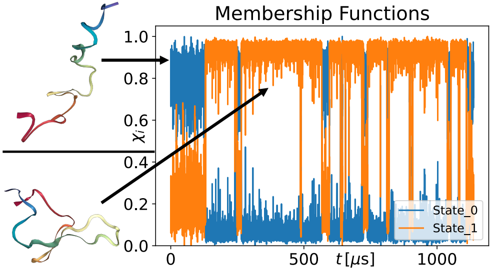
- Multiple eigenvalues: If φ_1 and φ_2 eigenfunctions with the same eigenvalue λ:

$$\frac{\varphi_1(x_k)}{\varphi_2(x_k)} = \frac{\lambda^k\varphi_1(x_0)}{\lambda^k\varphi_2(x_0)} = \frac{\varphi_1(x_0)}{\varphi_2(x_0)} = c \quad \text{invariant}.$$

More applications

Folding kinetics of a 35-amino acid protein

$$\mathrm{d}X_t = \nabla V(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$



Wu, Nüske, Paul, Klus, Koltai, Noé. The Journal of Chemical Physics 2016
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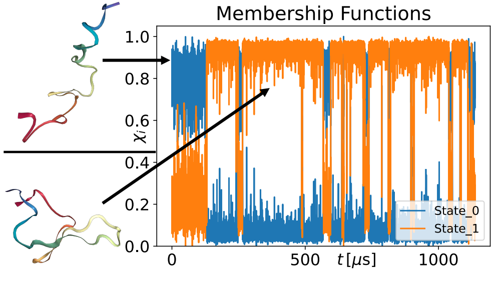
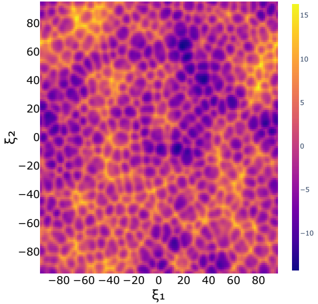


Kuramoto-Sivashinsky: Chaotic flames

$$\partial_t x + \nabla^2 x + \nabla^4 x + |\nabla x|^2 = 0.$$



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Stochastic Differential Equations

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Suitable assumptions: solution $(X_t)_{t\geq 0}$ exists and is **Markov process**.

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In the **deterministic case**, $\rho_t(x^0, A) = \delta_{x(t;x^0)}(A)$ such that $(\mathcal{K}\varphi)(x^0) = \varphi(x(t;x^0))$.

Extended Dynamic Mode Decomposition (EDMD)[2]

Consider **data points** $x_0, .., x_{d-1} \in \Omega$, a **dictionary** $\mathbb{V} := \operatorname{span}\{\{\psi_j\}_{j=1}^N\}$ with $\psi_j : \Omega \to \mathbb{R}$,

$$X := \left(\begin{pmatrix} \psi_1(x_0) \\ \vdots \\ \psi_N(x_0) \end{pmatrix} \Bigg| \dots \Bigg| \begin{pmatrix} \psi_1(x_{d-1}) \\ \vdots \\ \psi_N(x_{d-1}) \end{pmatrix} \right), \quad Y := \left(\begin{pmatrix} \psi_1(x^+(x_0)) \\ \vdots \\ \psi_N(x^+(x_0)) \end{pmatrix} \Bigg| \dots \Bigg| \begin{pmatrix} \psi_1(x^+(x_{d-1})) \\ \vdots \\ \psi_N(x^+(x_{d-1})) \end{pmatrix} \right).$$

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Data-based surrogate of the projected Koopman operator:

$$P_{\mathbb{V}} \mathcal{K}_{|\mathbb{V}} \approx \operatorname{argmin}_{K \in \mathbb{R}^{N \times N}} \|Y - KX\|_2^2$$

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Convergence in infinite-data and infinite-dictionary limit[3]

[3] Korda, Mezić, Journal of Nonlinear Science, 2018
[2] Williams et al., Journal of Nonlinear Science, 2015

Sampling strategies

Data $x_0, \ldots, x_{d-1} \in \Omega$ with successor states $y_0, \ldots, y_{d-1} \in \Omega$ from either

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   and that $(X_t)_{t\geq 0}$ is **ergodic**, i.e. for all $t > 0$

$$\rho_t(x, A) = 1\ \forall x \in A \quad \Rightarrow \quad \mu(A) \in \{0, 1\}$$

Theorem

For i.i.d. sampling w.r.t. μ, we have with $\varphi := \sum_{i=1}^{N} \psi_i^2 \in L^2_\mu$

$$\mathbb{P}\big(\|P_{\mathbb{V}}\mathcal{K}|_{\mathbb{V}} - K_d\|_F \leq \varepsilon\big) \gtrsim 1 - \frac{\|\varphi\|^2_{L^2_\mu}}{d\varepsilon^2}.$$

Philipp, S., Boshoff, Peitz, Nüske, Worthmann, arXiv:2402.02494, 2024.
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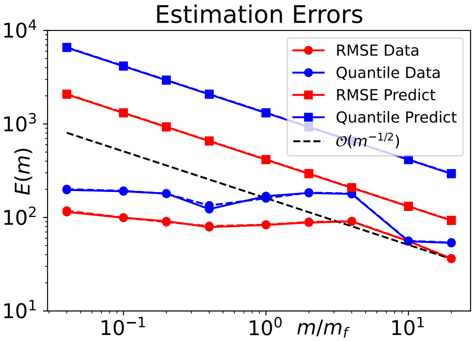
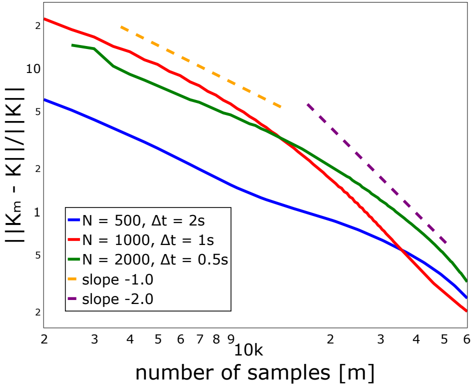
$\lambda = 1$ is never isolated for deterministic systems: Condition related to the spectral measure

Philipp, S., Boshoff, Peitz, Nüske, Worthmann, arXiv:2402.02494, 2024.
Nüske, Peitz, Philipp, S., Worthmann, Journal of Nonlinear Science, 2023
Kostic et al., NeurIPS, 2022/2023

Back to examples





Philipp, S., Boshoff, Peitz, Nüske, Worthmann, arXiv:2402.02494, 2024

Intuition: Monte-Carlo Integration

We may write the projected Koopman operator via[1]

$$P_{\mathbb{V}}\mathcal{K}|_{\mathbb{V}} = C^{-1}A \qquad \text{with} \qquad C_{i,j} = \langle \psi_i, \psi_j \rangle_{L^2(\Omega)}, \quad A_{i,j} = \langle \psi_i, \mathcal{K}\psi_j \rangle_{L^2(\Omega)}$$

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Data-driven approximation:

$$K_d = \operatorname{argmin}_{K \in \mathbb{R}^{N\times N}} \|Y - KX\|_2^2$$

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Data-driven approximation:

$$K_d = (XX^\top)^{-1}XY^\top = C_d^{-1}A_d$$

with

$$(C_d)_{i,j} = \tfrac{1}{d}\sum_{k=0}^{d-1}\psi_i(x_k)\psi_j(x_k), \quad (A_d)_{i,j} = \tfrac{1}{d}\sum_{k=0}^{d-1}\psi_i(x_k)\cdot(\mathcal{K}\psi_j)(x_k)$$

as

$$X := \left(\begin{pmatrix} \psi_1(x_0) \\ \vdots \\ \psi_N(x_0) \end{pmatrix} \middle| \dots \middle| \begin{pmatrix} \psi_1(x_{d-1}) \\ \vdots \\ \psi_N(x_{d-1}) \end{pmatrix} \right), \quad Y := \left(\begin{pmatrix} \psi_1(x^+(x_0)) \\ \vdots \\ \psi_N(x^+(x_0)) \end{pmatrix} \middle| \dots \middle| \begin{pmatrix} \psi_1(x^+(x_{d-1})) \\ \vdots \\ \psi_N(x^+(x_{d-1})) \end{pmatrix} \right).$$

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Intuition continued

It's all about $\frac{1}{d}\sum_{k=1}^{d-1}\psi_i(x_k)\psi_j(x_k) - \int \psi_i\psi_j$

Philipp, S., Boshoff, Peitz, Nüske, Worthmann, arXiv:2402.02494, 2024

Intuition continued

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Step 1: Estimate variance:

1. i.i.d. sampling $\sigma^2_{d,i,j} = \frac{1}{d}\sigma^2_{i,j}$
2. ergodic sampling $\sigma^2_{d,i,j} = \frac{1}{d}\Big(\underbrace{\sigma^2_{\infty,i,j}}_{\text{asymptotic variance}} - \underbrace{R^d_{i,j}}_{\text{remainder term}} \Big)$

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Philipp, S., Boshoff, Peitz, Nüske, Worthmann, arXiv:2402.02494, 2024

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Approximation error:

- **Projection**: $\mathcal{K} - P_{\mathbb{V}}\mathcal{K}_{|\mathbb{V}}$ due to finite dictionary $\mathbb{V} := \operatorname{span}\{(\psi_j)_{j=1}^N\}$
- **Estimation**: $P_{\mathbb{V}}\mathcal{K}_{|\mathbb{V}} - K_d$ due to finite data points $x_0, .., x_{d-1} \in \Omega$

Philipp, S., Boshoff, Peitz, Nüske, Worthmann, arXiv:2402.02494, 2024

Kernel Extended Dynamic Mode Decomposition

A **Reproducing Kernel Hilbert Space** $\mathbb{H}$ is a

- Hilbert space of functions $f : \Omega \to \mathbb{R}$
- with s.p.d. kernel $k : \Omega \times \Omega \to \mathbb{R}$ with $k(x, \cdot) \in \mathbb{H}$ for all $x \in \Omega$ and

$$\forall \varphi \in \mathbb{H} : \quad \varphi(x) = \langle \varphi, k(x, \cdot) \rangle$$ **reproducing property**

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kEDMD approximant

$$K_d = (XX^\top)^{-1} X^\top Y = X^{-1} X^{-1} XY = X^{-1} Y$$

Philipp, S., Worthmann, Peitz, Nüske, Applied and Computational Harmonic Analysis, 2024

Representation of the Koopman operator

Compatibility assumptions such that $\mathbb{H} \overset{d}{\hookrightarrow} L^2_\mu(\Omega)$

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$$\langle \mathcal{E}\psi, \eta\rangle = \int \psi(x)\langle k(x,\cdot),\eta\rangle\,d\mu(x) = \int \psi(x)\eta(x)\,d\mu(x) = \langle \psi,\eta\rangle_\mu \qquad \forall \psi \in L^2_\mu(\Omega), \eta \in \mathbb{H}$$

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Adjoint $\mathcal{E}^* : \mathbb{H} \to L^2_\mu(\Omega)$ is the compact inclusion operator from $\mathbb{H}$ into $L^2_\mu(\Omega)$

$$\mathcal{E}^*\eta = \eta, \qquad \eta \in \mathbb{H}.$$

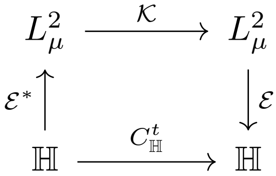
Koopman through the lens of the Mercer operator

Set

$$C_{\mathbb{H}}^t = \mathcal{E}\mathcal{K}\mathcal{E}^*$$

such that

$$\langle \eta, C_{\mathbb{H}}^t \psi \rangle = \langle \eta, \mathcal{K}\psi \rangle_\mu$$



Philipp, S., Worthmann, Peitz, Nüske, Applied and Computational Harmonic Analysis, 2024

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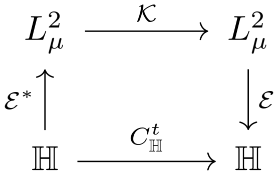

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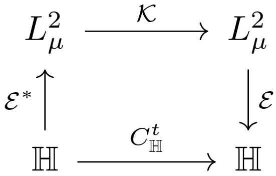

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Strategy: Approximate $C_{\mathbb{H}}^{-1}$, $C^t_{\mathbb{H}}$ and transfer to Koopman operator

Philipp, S., Worthmann, Peitz, Nüske, Applied and Computational Harmonic Analysis, 2024

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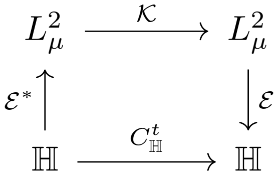

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Caution: $\mathcal{E}^{-1}, \mathcal{E}^{-*}$ are unbounded

Philipp, S., Worthmann, Peitz, Nüske, Applied and Computational Harmonic Analysis, 2024

Empirical estimator for $C^t_{\mathbb{H}}$

We compute

$$C^t_{\mathbb{H}}\psi = \mathcal{E}\mathcal{K}\mathcal{E}^*\psi = \int (\mathcal{K}\psi)(x)k(x,\cdot)\,d\mu(x) = \int\int \underbrace{\psi(y)k(x,\cdot)}_{=:C_{xy}\psi}\rho_t(x,dy)d\mu(x).$$

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Empirical estimator of time-lagged cross-covariance:

$$\hat{C}^{d,t}_{\mathbb{H}} := \frac{1}{d}\sum_{k=0}^{d-1} C_{x_k,y_k} \qquad \text{with matrix rep.} \qquad \frac{1}{d}(Y)_{ij} = k(x_i,y_j)$$

Empirical estimator for $C_{\mathbb{H}}^t$

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Empirical estimator of kernel covariance:

$$C_{\mathbb{H}} = \mathcal{E}^*\mathcal{E} = \int C_{xx}\,\mathrm{d}\mu(x) \approx \frac{1}{d}\sum_{i=0}^{d-1} C_{x_k,x_k} =: \hat{C}_{\mathbb{H}}^d \qquad \text{with matrix rep.} \qquad \frac{1}{d}(X)_{ij} = k(x_i, x_j)$$

Finite-data error bound

Theorem

For all $\varepsilon > 0$*, there is* $d_0 \in \mathbb{N}$ *such that for all* $d \geq d_0$

$$\mathbb{P}(\|C^t_{\mathbb{H}} - \hat{C}^{d,t}_{\mathbb{H}}\|_{HS} > \varepsilon) \leq \frac{\mathbb{E}_0(t) + R(d)}{d\varepsilon^2},$$

Philipp, S., Worthmann, Peitz, Nüske, Applied and Computational Harmonic Analysis, 2024
Mollenhauer, Klus, Schütte, Koltai, Journal of Machine Learning Research, 2022.

Main tool: Variance representations[1]

$$\mathbb{E}\left[\|\widehat{C}_{\mathbb{H}}^{d,t} - C_{\mathbb{H}}^{t}\|_{HS}^{2}\right] = \frac{1}{d}\Bigg[\mathbb{E}_0(t) + \underbrace{2\sum_{k=1}^{d-1} \tfrac{d-k}{d} \cdot \mathbb{E}\left[\langle C_{z_k} - C_{\mathbb{H}}^{t}, C_{z_0} - C_{\mathbb{H}}^{t}\rangle_{HS}\right]}_{R(d)}\Bigg],$$

[2] Philipp, S., Boshoff, Peitz, Nüske, Worthmann, arXiv:2402.02494, 2024

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For i.i.d. sampling, $R(d) \equiv 0$.

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For ergodic sampling, if 1 is an isolated eigenvalue of $\mathcal{K}$, then with $\mathcal{K}_0 = \mathcal{K}|_{\mathbb{1}^\perp}$

$$R(d) \leq 8\mathbb{E}_0(t)\|(I - \mathcal{K}_0)^{-2}\|$$

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Similar variance representations for arbitrary dictionaries[2]

[2] Philipp, S., Boshoff, Peitz, Nüske, Worthmann, arXiv:2402.02494, 2024
[1] Philipp, S., Worthmann, Peitz, Nüske, Applied and Computational Harmonic Analysis, 2024

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Then, for $\psi \in \mathbb{H}$,

$$\mathcal{K}\psi = \sum_{j=1}^{\infty} \langle \mathcal{K}\psi, e_j \rangle_\mu e_j = \underbrace{\sum_{j=1}^{N} \langle \mathcal{K}\psi, e_j \rangle_\mu e_j}_{=:\mathcal{K}_N \psi} + \sum_{j=N+1}^{\infty} \langle \mathcal{K}\psi, e_j \rangle_\mu e_j$$

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$$\mathcal{K}\psi = \sum_{j=1}^{\infty} \langle \mathcal{K}\psi, e_j \rangle_\mu e_j = \underbrace{\sum_{j=1}^{N} \langle \mathcal{K}\psi, e_j \rangle_\mu e_j}_{=:\mathcal{K}_N\psi} + \sum_{j=N+1}^{\infty} \langle \mathcal{K}\psi, e_j \rangle_\mu e_j$$

and we approximate with $\hat{e}_j = (\hat{\lambda}_j^{-1/2} \hat{f}_j)$ are eigenfunctions of $\hat{C}_{\mathbb{H}}^d = \frac{1}{d} \sum_{k=0}^{m-1} C_{x_k, x_k}$

$$\mathcal{K}_N\psi = \sum_{j=1}^{N} \langle C_{\mathbb{H}}^t \psi, e_j \rangle_{\mathbb{H}} e_j \approx \sum_{j=1}^{N} \langle \hat{C}_{\mathbb{H}}^{t,d} \psi, \hat{e}_j \rangle_{\mathbb{H}} e_j =: \hat{\mathcal{K}}_N^d \psi$$

Transfering this to Koopman operator

We have $C^t_{\mathbb{H}} = \mathcal{E}\mathcal{K}\mathcal{E}^*$, but $\mathcal{E}^{-1}, \mathcal{E}^{-*}$ are unbounded.

Remedy: Mercer basis of eigenfunctions of trace class operator $\mathcal{C}_{\mathbb{H}} = \mathcal{E}\mathcal{E}^*$ (f_j, λ_j), $\lambda_j \to 0$ with

$$(f_j) \text{ is ONB of } \mathbb{H} \qquad \text{and} \qquad (e_j) = (\lambda_j^{-1/2} f_j) \text{ is ONB of } L^2_\mu$$

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Back to kEDMD: $\hat{K}^d_N$ corresponds to K_d via $K_d = X^{-1}Y \approx X^\dagger_N Y$ with a rank-N truncation of X.

Main result: Estimation error

Theorem

Let $N \in \mathbb{N}$ and assume the gap condition

$$\gamma_N := \min_{j=1,\dots,N} \frac{\lambda_j - \lambda_{j+1}}{2} > 0.$$

Then, for each error bound $\varepsilon \in (0, \delta_N)$ and probabilistic tolerance $\delta \in (0,1)$ and

$$d \geq \max\{N, \frac{\mathbb{E}_0(t) + R(d)}{\varepsilon^2 \delta}\}$$

we have with probability at least $1-\delta$

$$\|\mathcal{K}_N - \hat{\mathcal{K}}_N^m\|_{\mathbb{H}\to L^2_\mu(\Omega)} \lesssim \left(\frac{1}{\sqrt{\lambda_N}} + \frac{N+1}{\gamma_N \lambda_N}\right)\varepsilon.$$

Invariance of the RKHS: Projection error

Corollary

Assume in addition that $\|\mathcal{K}_{\mathbb{H}}\| < \infty$. *Then for* $N \in \mathbb{N}$, $\varepsilon > 0$, $\delta \in (0,1)$, *there is* $m_0 \in \mathbb{N}$ *such that for all* $d \geq d_0$

$$\|\mathcal{K} - \hat{\mathcal{K}}_N^d\|_{\mathbb{H}\to L^2_\mu(\Omega)} \lesssim \left[\frac{1}{\sqrt{\lambda_N}} + \frac{N+1}{\gamma_N \lambda_N}\right]\varepsilon + \sqrt{\lambda_{N+1}}\,\|K\|_{\mathbb{H}\to\mathbb{H}}.$$

From L^2 to L^∞ bounds

With $P_{V_\mathcal{X}}$ $\mathbb{H}$-orthogonal projection onto $V_\mathcal{X} = \operatorname{span}\{k(x_i,\cdot), i = 0,\ldots,d-1\}$

$$\mathcal{K} \approx K_d := P_{V_\mathcal{X}} \mathcal{K} P_{V_\mathcal{X}} \quad \rightsquigarrow \quad \text{matrix representation} \quad K_X^{-1} K_{X,X^+}$$

with

$$(K_X)_{ij} = k(x_i, x_j), \qquad (K_{X,X^+})_{ij} = k(x_i, F(x_j)).$$

Wendland, Advances in Computational Mathematics 1995

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Here: compactly supported radially symmetric Wendland kernel:

$$\mathbb{H} \cong H^{\sigma(p)}(\Omega) \qquad p \sim \text{smoothness}$$

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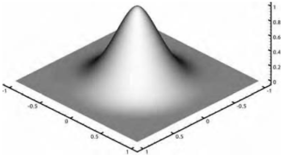

Projection error $P_{V_\mathcal{X}} - I$ controlled by fill distance

$$h_\mathcal{X} := \sup_{x\in\Omega} \min_{1\le i\le d} \|x - x_i\|_2$$

Wendland, Advances in Computational Mathematics 1995

Error bound on $\mathcal{K} - P_{V_\mathcal{X}} \mathcal{K} P_{V_\mathcal{X}}$

Theorem[2]

If flow is in C^m, then, for $\sigma(p) \leq m$, $\mathcal{K} H^{\sigma(p)}(\Omega) \subset H^{\sigma(p)}(\Omega)$, and, for $f \in H^{\sigma(p)}(\Omega)$,

$$\|\mathcal{K}f - K_d f\|_\infty \lesssim h_\mathcal{X}^{p+1/2} \|f\|_{H^{\sigma(p)}(\Omega)}.$$

[2] Köhne, Philipp, S., Schiela, Worthmann, SIADS 2025

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Theorem (Wendland 1995)

There are $C, h_0 > 0$ such that for every set $\mathcal{X} = \{x_i\}_{i=1}^d \subset \Omega$ with $h_\mathcal{X} \leq h_0$ and all $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq k$,

$$\|I - P_{V_\mathcal{X}}\|_{\mathbb{H}_{\Phi_{n,k}}\to C_b(\Omega)} \leq C h_\mathcal{X}^{k+1/2}.$$

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Proof pt. 2

$$\left\|\mathcal{K} - K_d\right\|_{\mathbb{H}\to C_b} = \left\|\mathcal{K} - P_{V_\mathcal{X}}\mathcal{K}P_{V_\mathcal{X}}\right\|_{\mathbb{H}\to C_b} \lesssim (1 + \|\mathcal{K}\|_{\mathbb{H}\to\mathbb{H}})\|I - P_{V_\mathcal{X}}\|_{\mathbb{H}\to C_b}$$

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Let $\mathbb{H}$ be a Gaussian RKHS on $\Omega = \mathbb{R}^{n_x}$. Then $\mathcal{K}\mathbb{H} \subset \mathbb{H}$ if and only if the flow is affine-linear, i.e., $x(t; x^0) = A(t)x^0 + b(t)$.

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The Koopman operator on RKHS is always closed.

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Proposition[4]

The Koopman operator on RKHS is always closed. In particular, $\mathcal{K}\mathbb{H} \subset \mathbb{H}$ implies $\|\mathcal{K}\| < \infty$.

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Koopman is closed

Adjoint-like property:

$$\langle \mathcal{K}\varphi, k(x,\cdot)\rangle = \varphi(F(x)) = (\mathcal{K}\varphi)(x) = \langle \varphi, k(F(x),\cdot)\rangle$$

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Corollary[5]

If flow F is in C^m, then for Wendland kernels with $\mathbb{H} = H^{\sigma(p)}(\Omega)$, $\sigma(p) \le m$,

$$\|\mathcal{K}\|_{\mathbb{H}\to\mathbb{H}} < \infty$$

Sketch of the proof: Chain rule, as $\mathcal{K}\varphi = \varphi \circ F$.

[5] Köhne, Philipp, S., Schiela, Worthmann, SIADS 2025

EDMD for control systems

Consider the control-affine system

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No linearity to be expected for nonlinear liftings.

Bilinear surrogates

Bilinear surrogate model [Williams et al. '16, Surana '16, Peitz et al. '20]

Let $u \in \mathbb{R}^{n_u}$ and consider the Koopman operator

$$(\mathcal{K}_u^t \varphi)(x^0) = \varphi(x(t; x^0, u))$$

Philipp, S., Worthmann, Peitz, Nüske, Journal of Nonlinear Science 2023, 2025

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Strongly continuous semigroup (in L^2 or C):

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hence $\mathcal{L}_u = \mathcal{L}_0 + \sum_{i=1}^{n_c} u_i(\mathcal{L}_{e_i} - \mathcal{L}_0)$, such that

$$\dot{\varphi} = \mathcal{L}_u\varphi = \mathcal{L}_0 + \sum_{i=1}^{n_c} u_i(\mathcal{L}_{e_i} - \mathcal{L}_0)\varphi$$

Philipp, S., Worthmann, Peitz, Nüske, Journal of Nonlinear Science 2023, 2025

EDMD-based exponentially stabilizing controller

Approximately bilinear system

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Strässer, S., Worthmann, Berberich, Allgöwer, IEEE TAC 2025
Strässer, Worthmann, Mézic, Berberich, S. Allgöwer, submitted 2025

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$\hookrightarrow$ LMI-based tools from robust control to design state-feedback controller

$$\mu(x) = (I - L_w(\Lambda^{-1} \otimes \hat{\Phi}(x)))^{-1} L P^{-1} \hat{\Phi}(x)$$

ensuring exponential stability (with probability $1-\delta$) for all initial conditions in the safe operating region

$$\hat{x} \in \{x \in \mathbb{R}^n \mid \hat{\Phi}(x)^\top P^{-1} \hat{\Phi}(x) \le 1\},$$

where $P, L, L_w, \Lambda, \ldots$ solve two Linear Matrix Inequalities.

Strässer, S., Worthmann, Berberich, Allgöwer, IEEE TAC 2025
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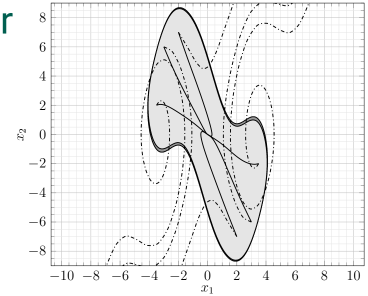
EDMD-based exponentially stabilizing controller

Nonlinear inverted pendulum

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = \frac{g}{l}\sin(x_1(t)) - \frac{b}{ml^2}x_2(t) + \frac{1}{ml^2}u(t)$$

with mass m, length l, rotational friction coefficient b, and gravitational constant g.



Nonlinear system

$$\dot{x}_1(t) = \rho x_1(t),$$
$$\dot{x}_2(t) = \lambda(x_2(t) - x_1(t)^2) + u(t)$$

with $\rho, \lambda \in \mathbb{R}$