
Operator Learning

Learning forward operator

Nicolas Boullé

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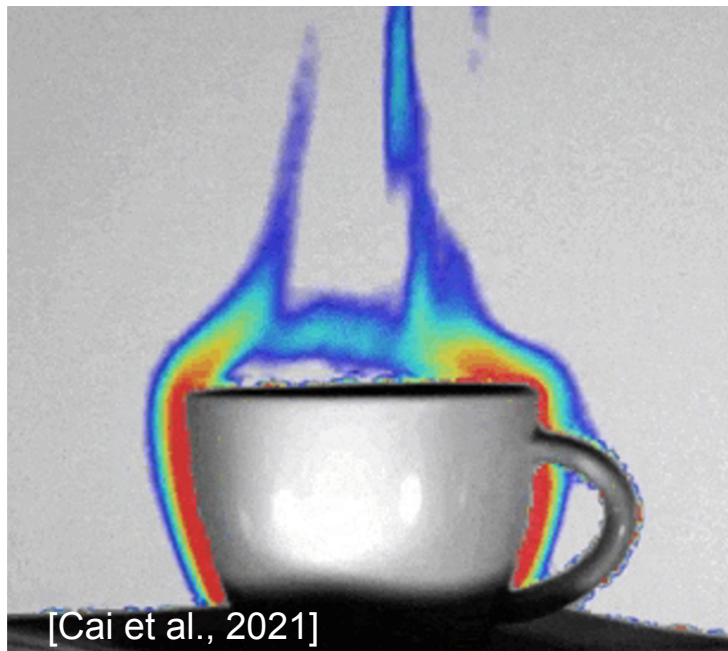
IMPERIAL

Three areas of SciML

Physics-informed machine learning

George Em Karniadakis^{1,2,3}, Ioannis G. Kevrekidis^{5,4}, Lu Lu⁵, Paris Perdikaris⁶, Sifan Wang⁷ and Liu Yang¹

Abstract | Despite great progress in simulating multiphysics problems using the numerical discretization of partial differential equations (PDEs), one still cannot seamlessly incorporate noisy data into existing algorithms, mesh generation remains complex, and high-dimensional problems governed by parameterized PDEs cannot be tackled. Moreover, solving inverse problems with hidden physics is often prohibitively expensive and requires different formulations and elaborate computer codes. Machine learning has emerged as a promising alternative, but training deep neural networks requires big data, not always available for scientific problems. Instead, such networks can be trained from additional information obtained by enforcing the physical laws (for example, at random points in the continuous space-time domain). Such physics-informed learning integrates (noisy) data and mathematical models, and implements them through neural networks or other kernel-based regression networks. Moreover, it may be possible to design specialized network architectures that automatically satisfy some of the physical invariants for better accuracy, faster training and improved generalization. Here, we review some of the prevailing trends in embedding physics into machine learning, present some of the current capabilities and limitations and discuss diverse applications of physics-informed learning both for forward and inverse problems, including discovering hidden physics and tackling high-dimensional problems.

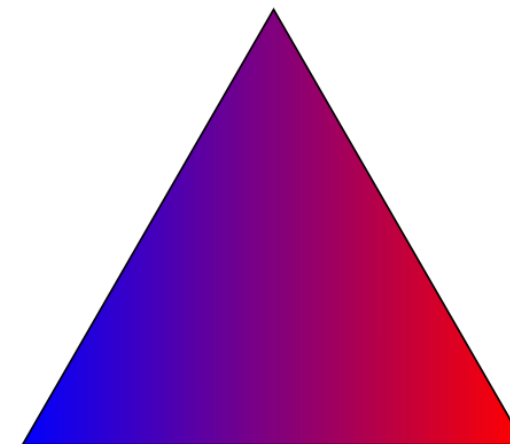


Inverse Problem



Forward Problem

PDE Discovery



PDE Solvers

Operator Learning



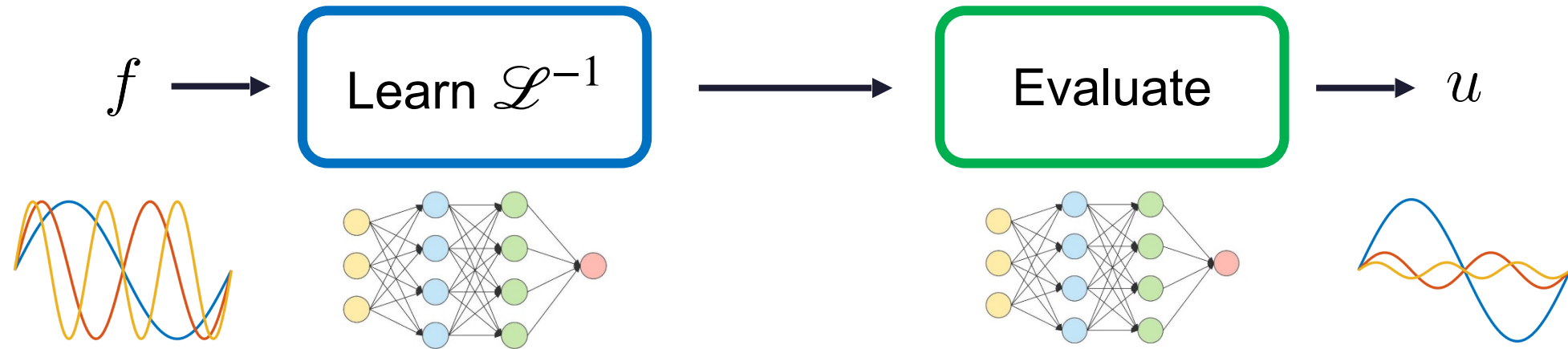
Small data
Lots of physics

Some data
Some physics

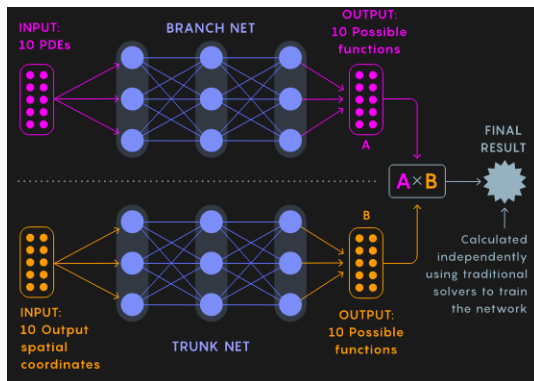
Lots of data
No physics

Operator learning

Aim: Approximate solution operators of unknown PDEs $\mathcal{L}(u) = f$

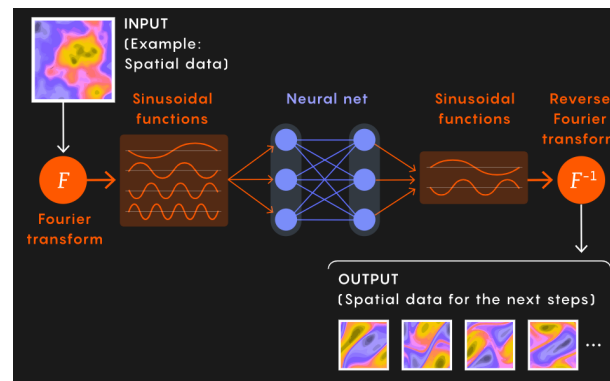


DeepONet



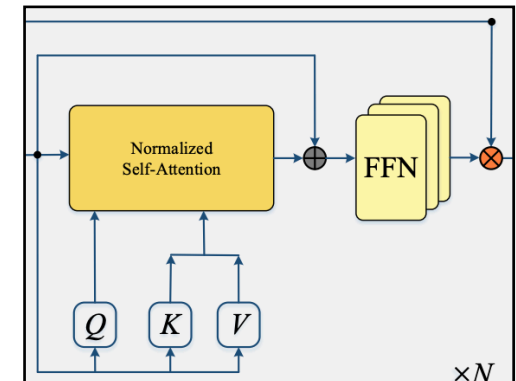
[Quanta Magazine; Lu et al, 2021]

Fourier Neural Operator



[Quanta Magazine; Li et al, 2020]

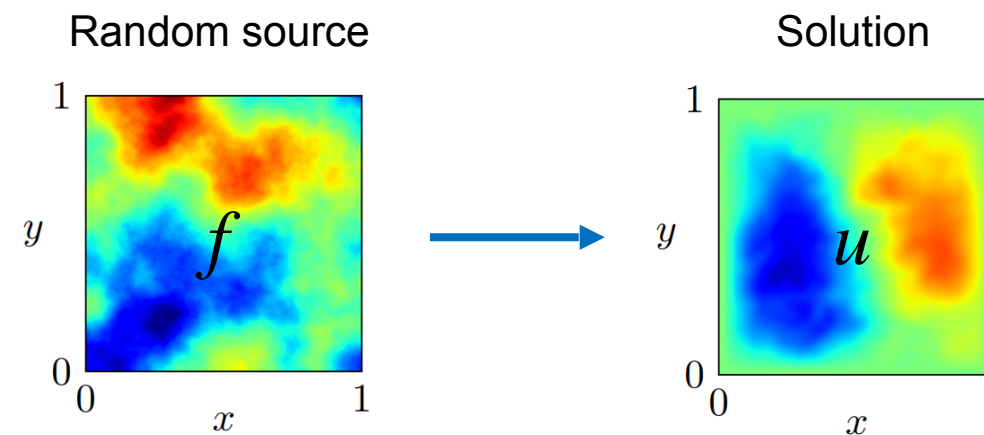
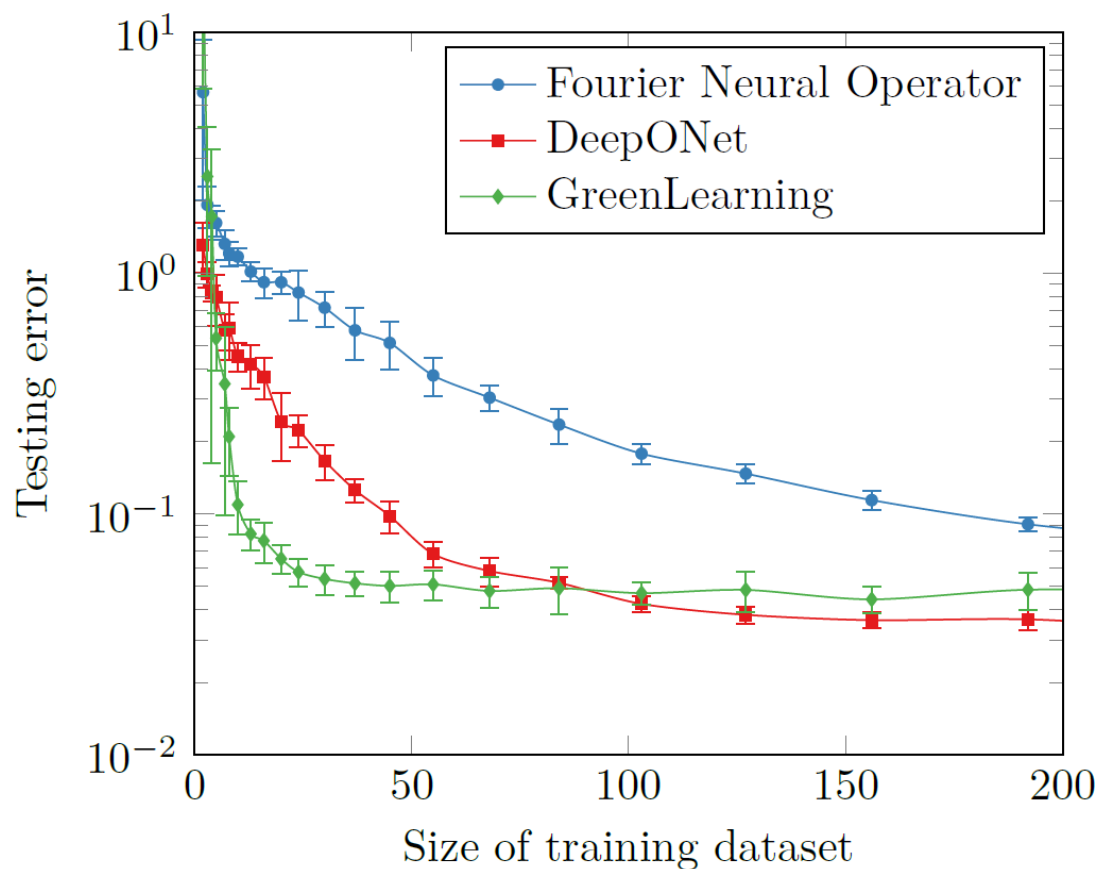
GNOT



[Hao et al., 2023]

Sample complexity in operator learning

How many pairs (f, u) are needed to approximate $\mathcal{L}^{-1}$?



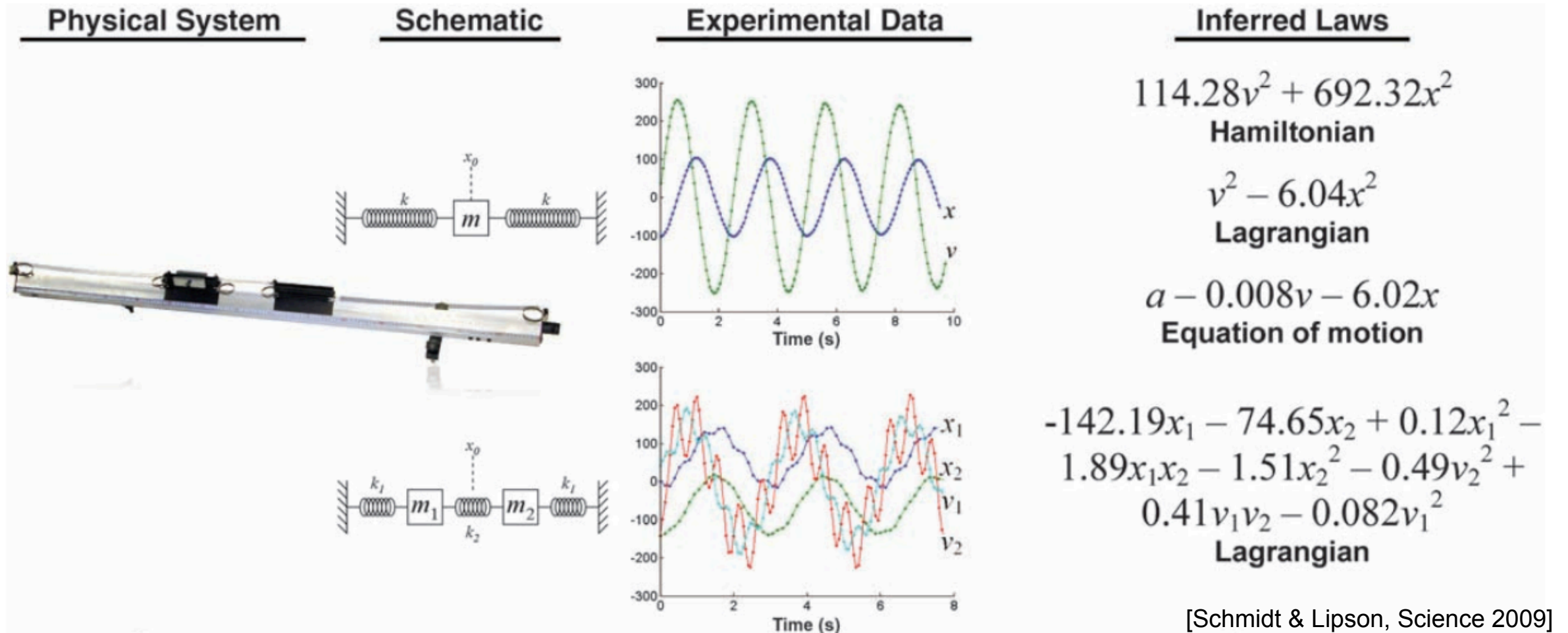
Theorem (B., Halikias, Townsend, 2023).

*There is a randomized algorithm that converges **exponentially fast** to solution operators of elliptic PDEs with respect to the number of pairs.*

Downside: black-box operator difficult to interpret

PDE Discovery

Aim: Find a **symbolic expression** for the forward operator $\mathcal{L}(u) = f$



[Schmidt & Lipson, Science 2009]

PDE Discovery

Aim: Find a **symbolic expression** for the forward operator

$$\mathcal{L}(u) = f$$

SCIENCE ADVANCES | RESEARCH ARTICLE

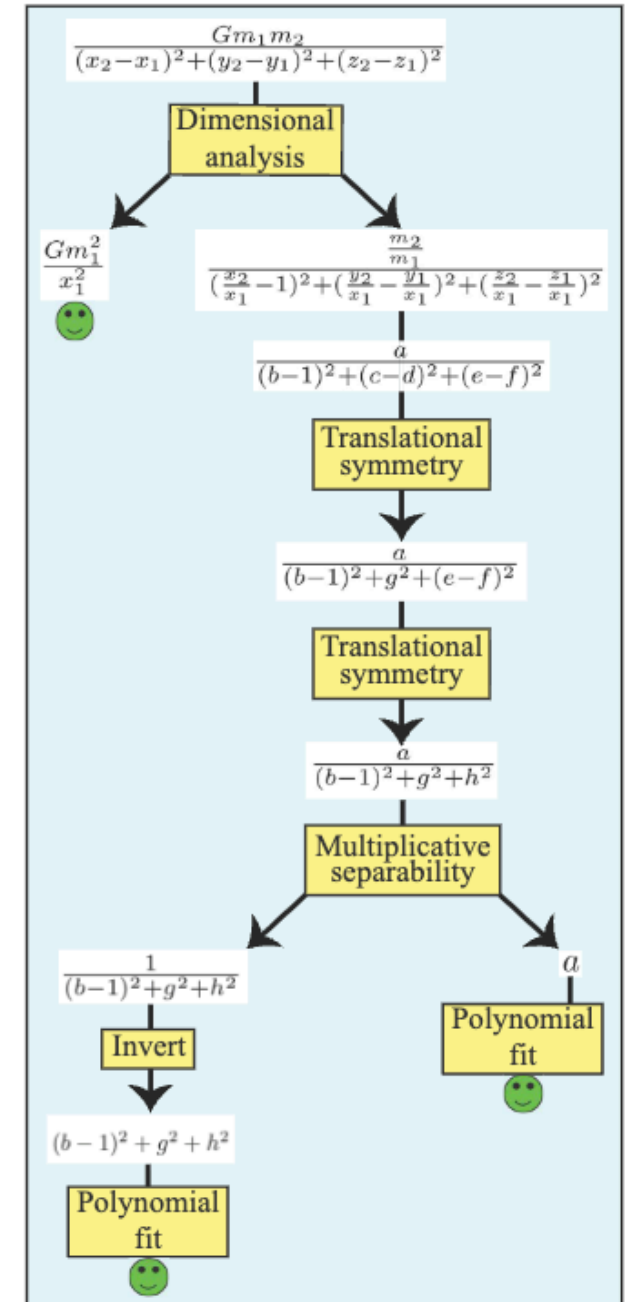
COMPUTER SCIENCE

AI Feynman: A physics-inspired method for symbolic regression

Silviu-Marian Udrescu¹ and Max Tegmark^{1,2*}

A core challenge for both physics and artificial intelligence (AI) is symbolic regression: finding a symbolic expression that matches data from an unknown function. Although this problem is likely to be NP-hard in principle, functions of practical interest often exhibit symmetries, separability, compositionality, and other simplifying properties. In this spirit, we develop a recursive multidimensional symbolic regression algorithm that combines neural network fitting with a suite of physics-inspired techniques. We apply it to 100 equations from the *Feynman Lectures on Physics*, and it discovers all of them, while previous publicly available software cracks only 71; for a more difficult physics-based test set, we improve the state-of-the-art success rate from 15 to 90%.

Given data $\{x_1, \dots, x_n, y\}$, the aim is to discover an **unknown** function f satisfying $y = f(x_1, \dots, x_n)$



PDE Discovery

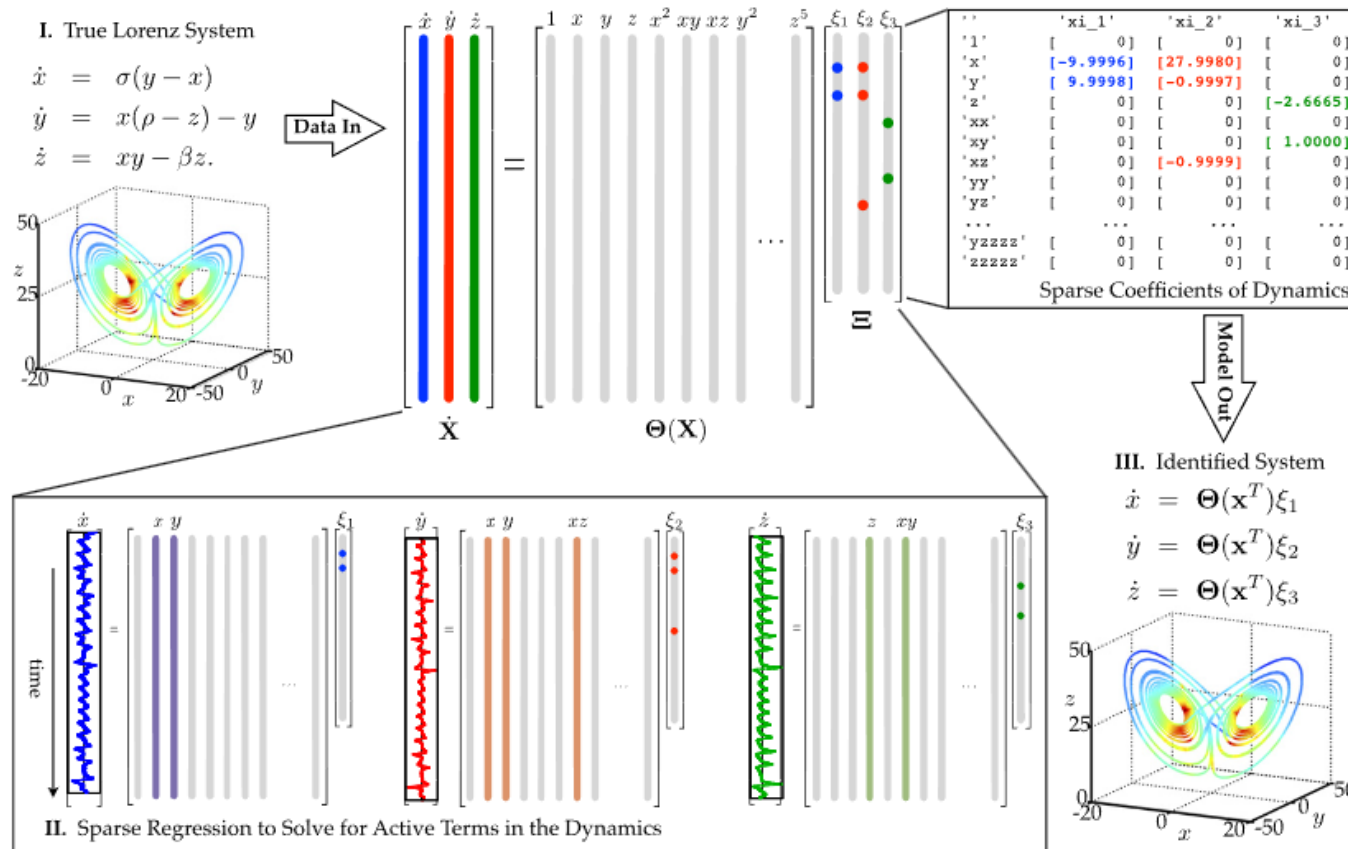
Sparse Identification of Nonlinear Dynamics (SINDy)



Steven Brunton



Nathan Kutz



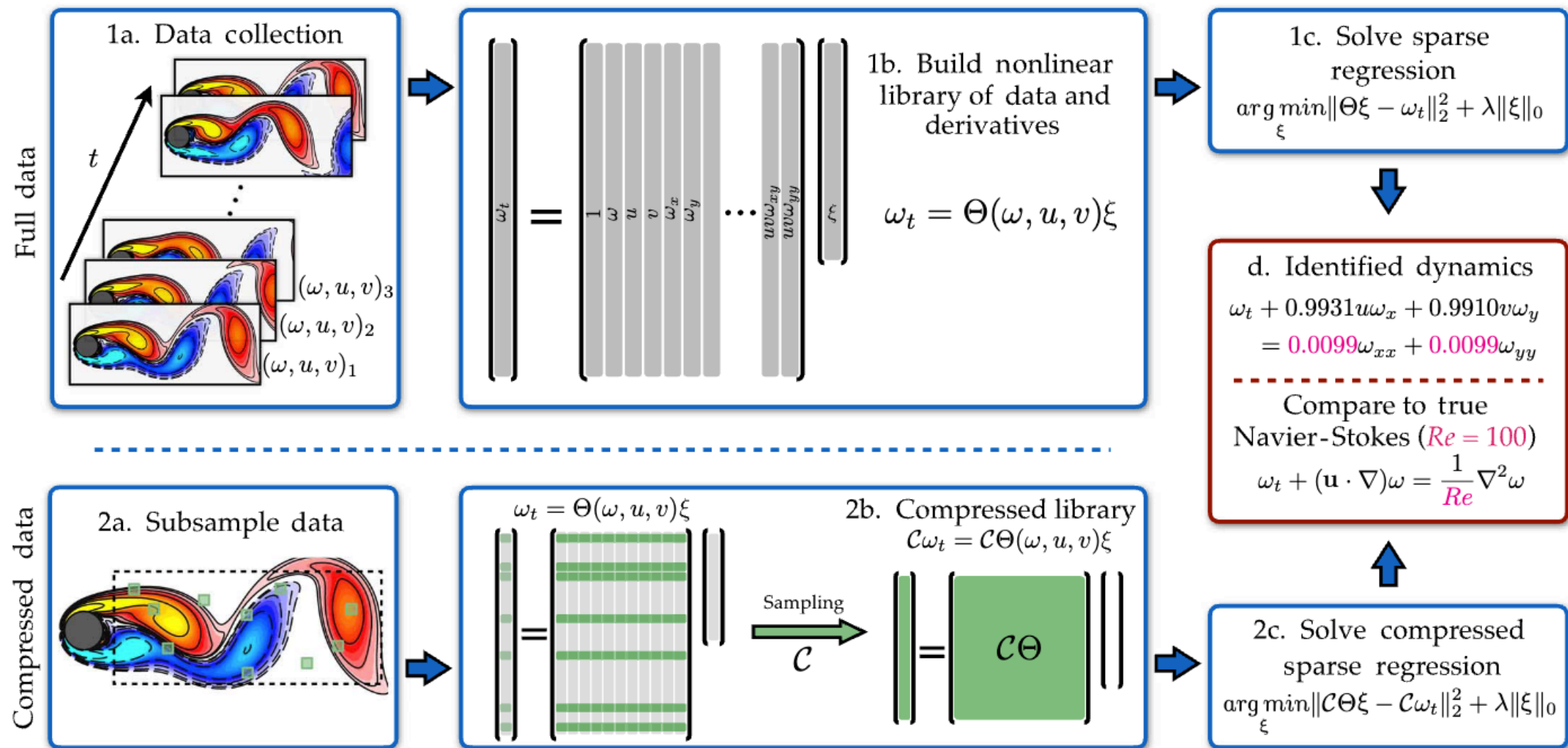
[Brunton et al, 2016]

1. Observation of trajectory $X(t)$
2. Choice of dictionary $\Theta(X)$
3. Solve least-square problem

$$\Theta(X)C = \dot{X}$$

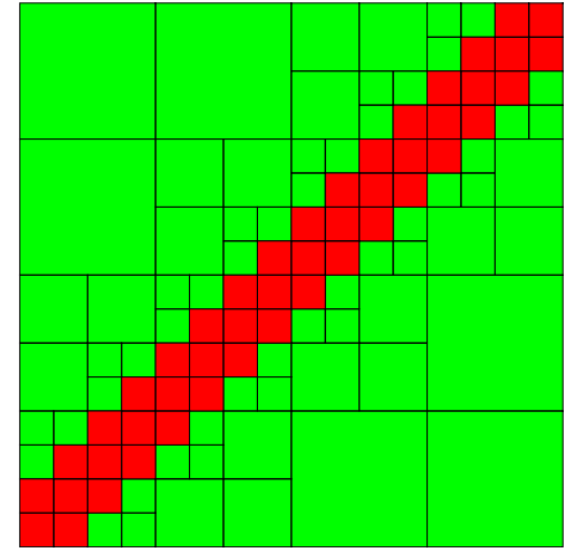
PDE Discovery

Data-driven discovery of partial differential equations



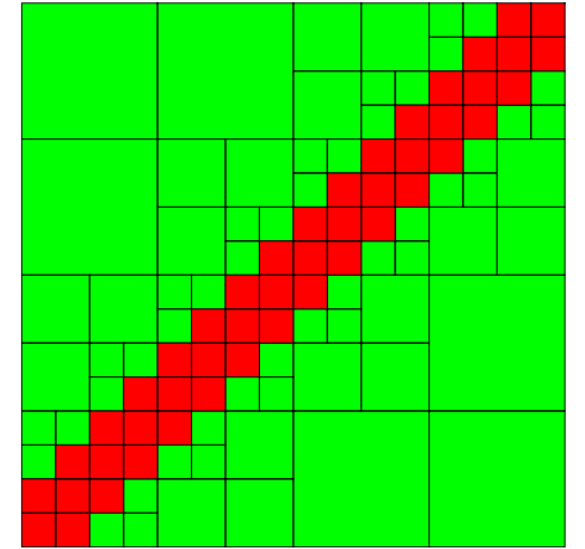
PDE Discovery

There have been several recent works on [sample complexity](#) in operator learning, based on numerical linear algebra ideas [B. & Townsend, 2024]



PDE Discovery

There have been several recent works on **sample complexity** in operator learning, based on numerical linear algebra ideas [B. & Townsend, 2024]



Comparatively, **fewer guarantees** are known for SINDy-type algorithms

1. **How much** data do we need to identify ODEs / PDEs?
2. What **type** of data do we need?

Guarantees for system identification

For Differential Equations with r Parameters, $2r + 1$ Experiments Are Enough for Identification

E. D. Sontag*

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<http://www.math.rutgers.edu/~sontag>

e-mail: sontag@hilbert.rutgers.edu

Conditions for uniqueness

PROBLEM 2.7 (Global identifiability).

IN: Σ : an algebraic differential model given by rational functions

$f(x, \mu, u)$ and $g(x, \mu, u)$

θ^ℓ : a subset of $\theta = \mu \cup x^*$ such that every parameter in θ^ℓ is locally identifiable

OUT: θ^g : the set of all globally identifiable parameters in θ^ℓ

[Hong et al., 2020]

Distinguishing between parameters

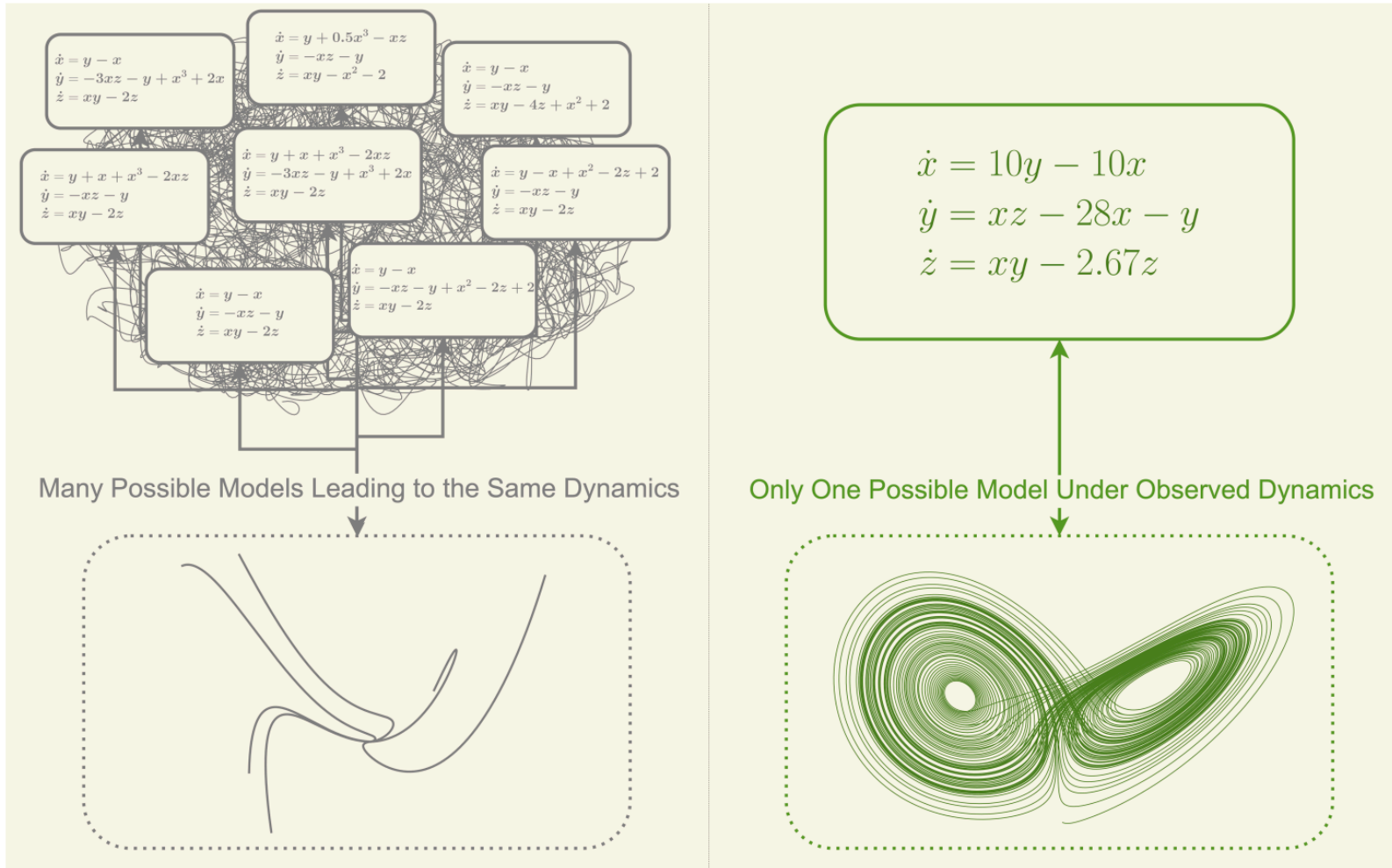
PDE class	Equivalent (E) or sufficient (S) for uniqueness
Linear PDE	(E) $g_1, \dots, g_k$ are linearly independent
Polynomial or algebraic PDE	(E) There is no hypersurface/algebraic set $A \neq \mathbb{R}^k$ such that $\mathcal{D} \subset A$ (E) $\mathcal{D}$ has a non trivial "correspondence I " (E) Assuming u is algebraic: $k \leq m + 1$ and Jacobian $J_g(t, x)$ has full rank for some $t, x \in U$
Analytic PDE	(E) There is no C-analytic set $A \neq \mathbb{R}^k$ such that $\mathcal{D} \subset A$ (S) There is no analytic set $A \neq \mathbb{R}^k$ such that $\mathcal{D} \subset A$ (S) $\lambda^k(\mathcal{D}) > 0$ (S) The Jacobian $J_g(t, x)$ is full-rank for some $t, x \in U$
$C^p, 0 \leq p \leq \infty$	(E) $\overline{\mathcal{D}} = \mathbb{R}^k$

[Scholl et al., 2026]

Identifiable parameters

Guarantees for system identification

Learn the map $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ from a trajectory $\dot{u} = F(u)$



Analytic systems identification

We consider the following system:

$$\sum_{i=1}^N c_i^* \mathcal{D}_i(u, f) = \mathcal{D}_0(u, f)$$

$\mathcal{D}_i : \mathcal{U} \times \mathcal{F} \rightarrow \mathcal{V}$ are **known analytic dictionary functions** between Banach spaces and $\mathbf{c}^* = (c_1^*, \dots, c_N^*) \in \mathbb{R}^N$ are **unknown** coefficients

Analytic systems identification

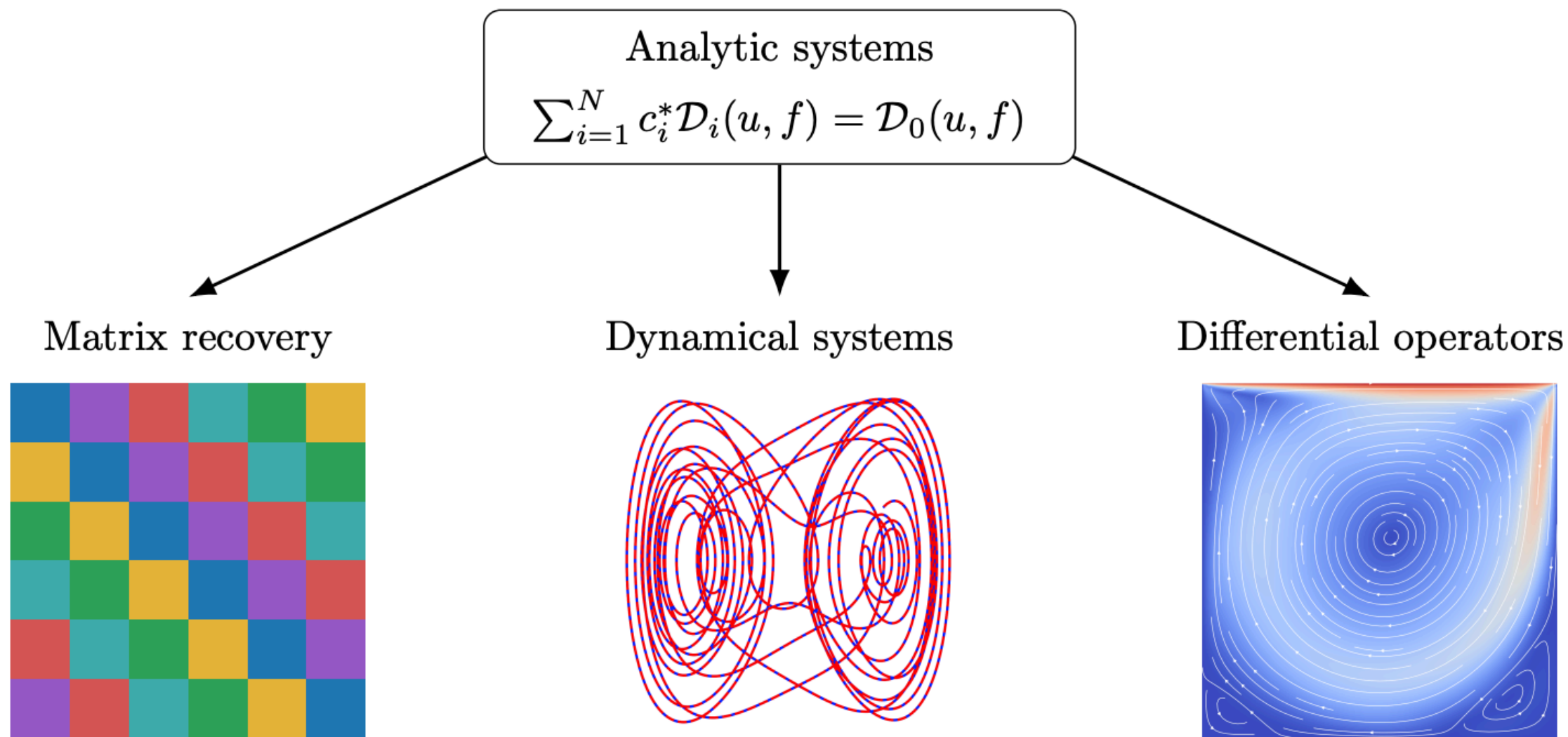
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When does a single pair (f, u) uniquely recovers the coefficients $\mathbf{c}^*$?

Analytic systems identification



Examples of analytic systems identification

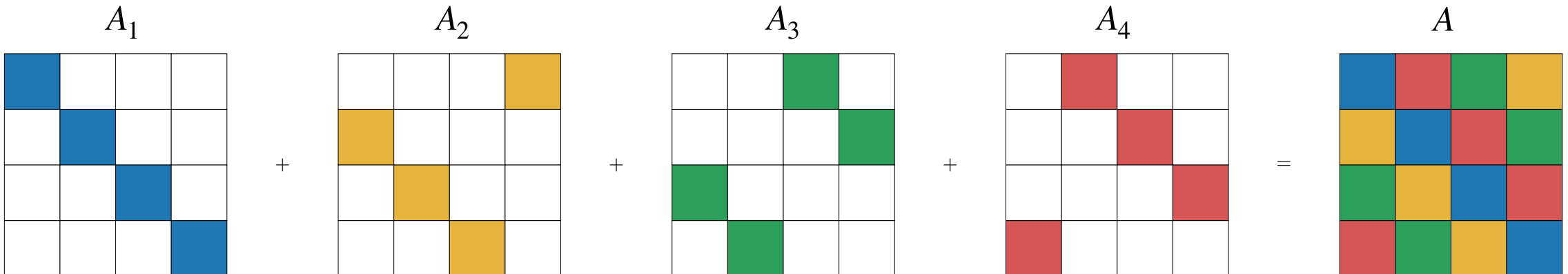
1. Matrix recovery

$$\sum_{i=1}^N c_i^* \mathcal{D}_i(u, f) = \mathcal{D}_0(u, f)$$

One seeks a **linearly parametrized matrix** of the form

$$u = Af = \sum_{i=1}^N c_i^* A_i f$$

Example: Circulant matrices



Examples of analytic systems identification

2. Dynamical systems

$$\sum_{i=1}^N c_i^* \mathcal{D}_i(u, f) = \mathcal{D}_0(u, f)$$

One seeks an **ordinary differential equation** of the form

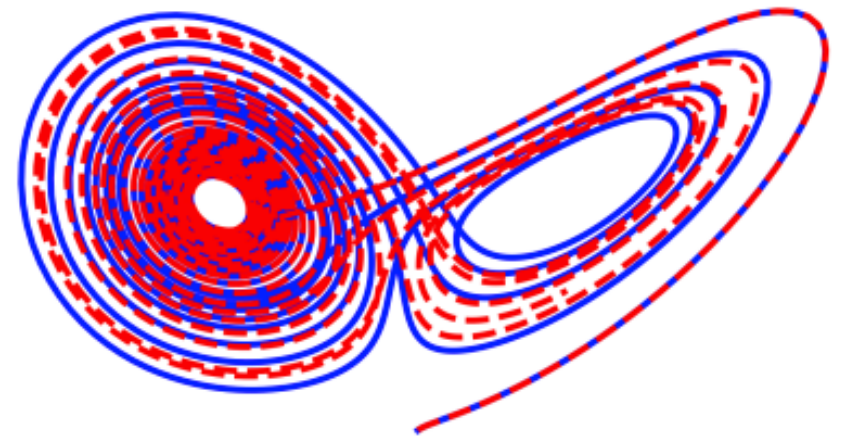
$$\dot{u}(t) = \sum_{i=1}^N c_i^* D_i(t, u, f)$$

$$u(0) = f_0$$

Example: Lorenz system

$$\mathcal{D}_0[u, (f_0, f)](t) = u - f_0$$

$$\mathcal{D}_i[u, (f_0, f)](t) = \int_0^t D_i(s, u(s), f(s)) ds$$



Examples of analytic systems identification

3. Boundary value problems

$$\sum_{i=1}^N c_i^* \mathcal{D}_i(u, f) = \mathcal{D}_0(u, f)$$

One seeks a **partial differential equation** of the form

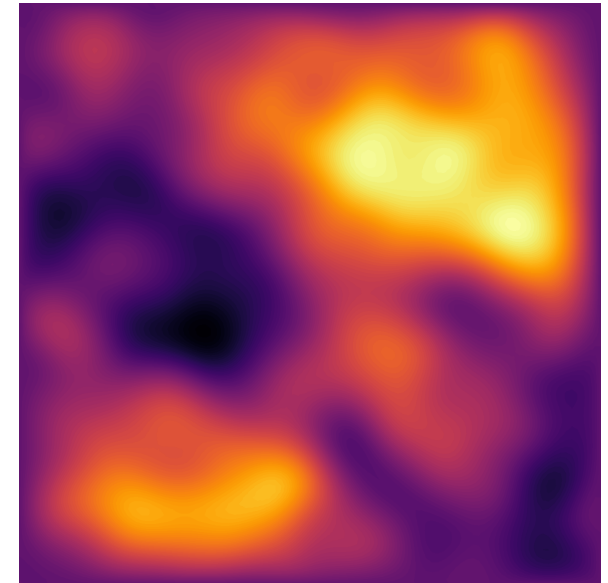
$$\sum_{i=1}^N c_i^* \mathcal{D}_i(u) = f$$

Example: Allen-Cahn equation $\frac{1}{4}\Delta u - 5u - u^3 + u^5 = f$

$$\mathcal{D}_0(u, f) = f$$

$$\mathcal{D}_1(u, f) = \Delta u \quad \mathcal{D}_3(u, f) = u^3$$

$$\mathcal{D}_2(u, f) = u \quad \mathcal{D}_4(u, f) = u^5$$



When is one-shot recovery possible?

We consider the following system:

$$\sum_{i=1}^N c_i^* \mathcal{D}_i(u, f) = \mathcal{D}_0(u, f)$$

When does a **single pair** (f, u) uniquely recovers the **coefficients** $\mathbf{c}^*$?

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When does a **single pair** (f, u) uniquely recovers the **coefficients** $\mathbf{c}^*$?

If $\mathcal{D}_i(u, f)$ are **linearly dependent**: $\sum_{i=1}^N a_i \mathcal{D}_i(u, f) = 0$ and recovery is **impossible**

Solution 1

$$\sum_{i=1}^N c_i^* \mathcal{D}_i(u, f) = \mathcal{D}_0(u, f)$$

Solution 2

$$\sum_{i=1}^N (c_i^* + a_i) \mathcal{D}_i(u, f) = \mathcal{D}_0(u, f)$$

Parameter-to-output map

We consider the following system:

$$\sum_{i=1}^N c_i^* \mathcal{D}_i(u, f) = \mathcal{D}_0(u, f)$$

Given (f, u) , we define the following **parameter-to-output** map:

$$L_{f,u} : \mathbf{c} \mapsto \sum_{i=1}^N c_i \mathcal{D}_i(u, f)$$

$L_{f,u}$ injective

Unique solution to $L_{f,u}[\mathbf{c}] = \mathcal{D}_0(u, f)$

$L_{f,u}$ non-injective

One-shot recovery is impossible

Degenerate input set

We assume that the solution operator $\mathcal{G} : f \mapsto u$ associated with

$$\sum_{i=1}^N c_i^* \mathcal{D}_i(u, f) = \mathcal{D}_0(u, f) \text{ is analytic and well-posed}$$

Set of **degenerate inputs** f such that one-shot recovery is **impossible**:

$$\mathcal{F}_{\text{deg}}(\mathbf{c}^*) = \{f \mid L_{f, \mathcal{G}(f)} \text{ injective}\}$$

If we select $f \in \mathcal{F}$, **how likely** is it to be a degenerate input?

Zero-one law for one-shot recovery

Theorem. [B., Halikias, Otto, Townsend 2026]

Exactly one of the following is true:

- If f is sampled from a Gaussian measure on $\mathcal{F}$ then $f \notin \mathcal{F}_{\text{deg}}(\mathbf{c}^*)$ with probability 1 (recovery is almost-surely possible)
- All inputs are degenerate: $\mathcal{F}_{\text{deg}}(\mathbf{c}^*) = \mathcal{F}$ (recovery is impossible)

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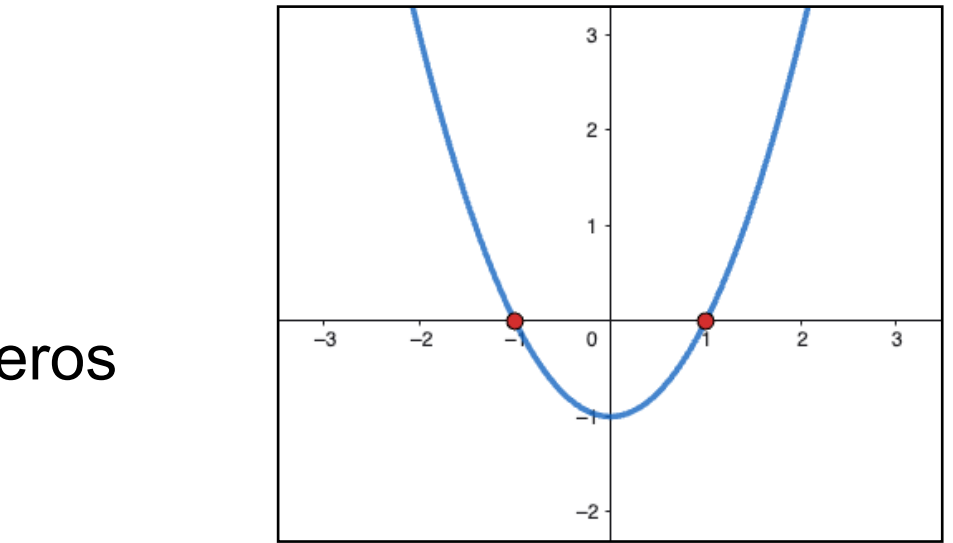
Proof.

Properties of zero-level sets of analytic functions

Zero-level sets of analytic functions

Dimension 1

If $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is non-zero and **analytic** then its zeros are **isolated**

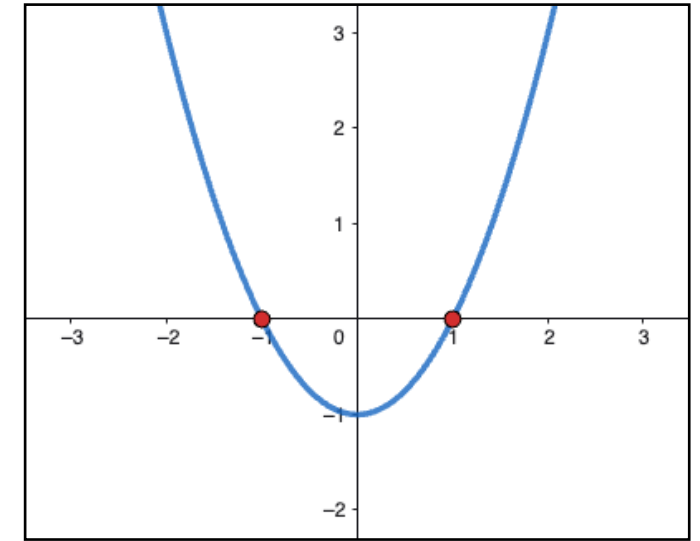


Otherwise all derivatives at an accumulation point would vanish hence the Taylor series and ϕ would be zero.

Zero-level sets of analytic functions

Dimension 1

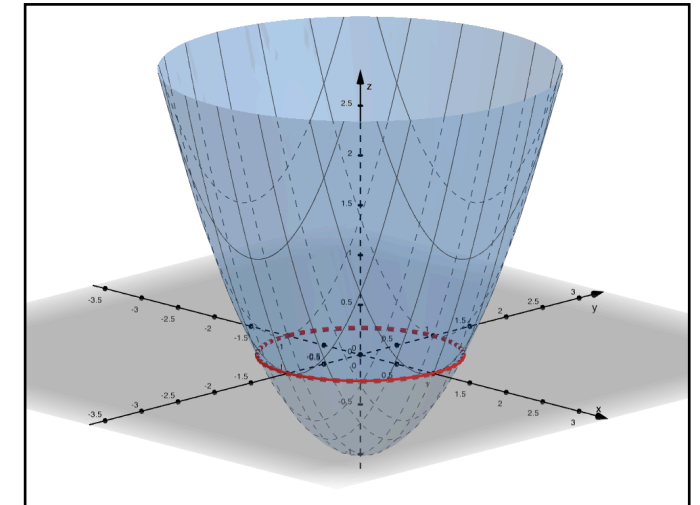
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Finite dimensions [e.g., Federer 1969]

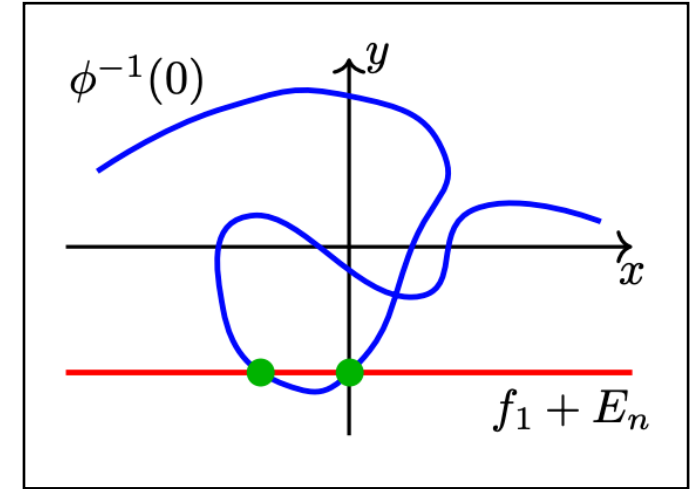
If $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is non-zero and **analytic** then $\phi^{-1}\{0\}$ has **Lebesgue measure 0** in $\mathbb{R}^n$



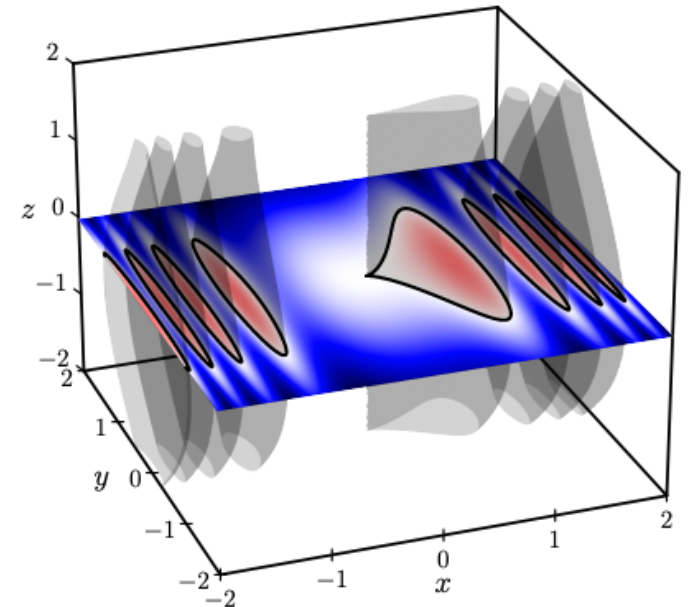
Zero-level sets of analytic functions

Infinite dimensions [B., Halikias, Otto, Townsend 2026]

If $\phi : \mathcal{F} \rightarrow \mathbb{R}$ is non-zero and **analytic** then its zero-level set $\phi^{-1}\{0\}$ is **compactly slice-null**



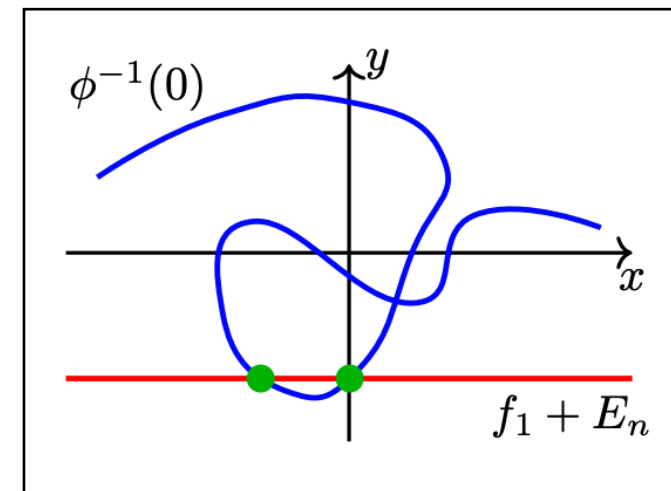
Its intersection with large enough finite-dimensional affine subspaces translated over any compact set has zero Lebesgue measure on each slice



Zero-level sets of analytic functions

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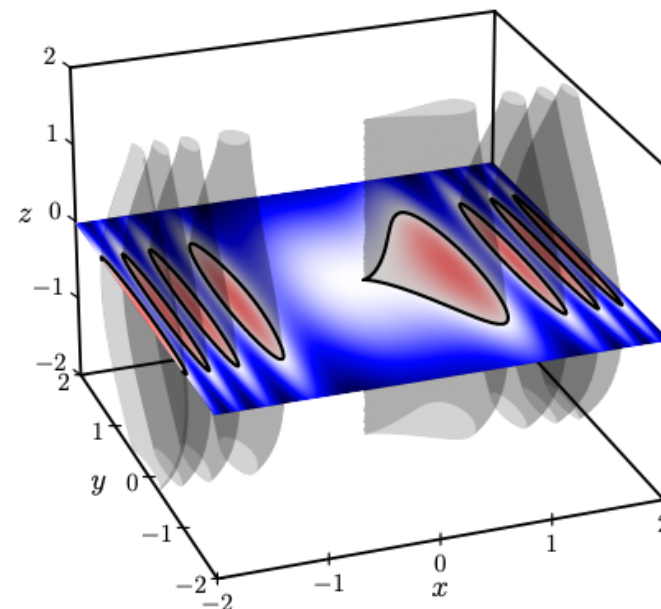
Its intersection with large enough finite-dimensional affine subspaces translated over any compact set has zero Lebesgue measure on each slice

Compactly slice-null implies **Gaussian measure zero**

$$\gamma(\phi^{-1}(0) \cap K_\varepsilon) = \int_Y \gamma^y(\phi^{-1}(0) \cap K_\varepsilon) \Pi_* \gamma(dy)$$

Disintegration theorem

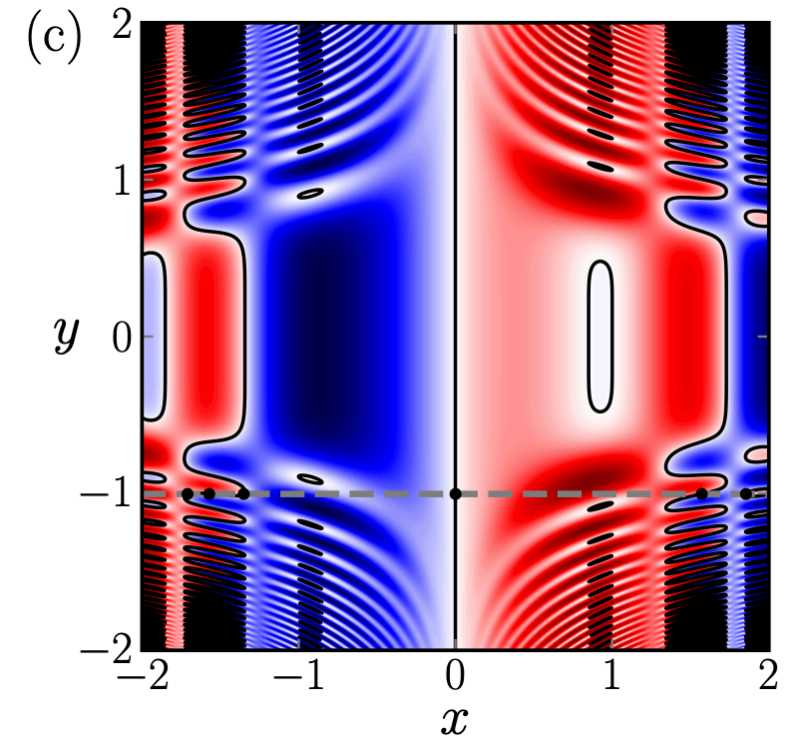
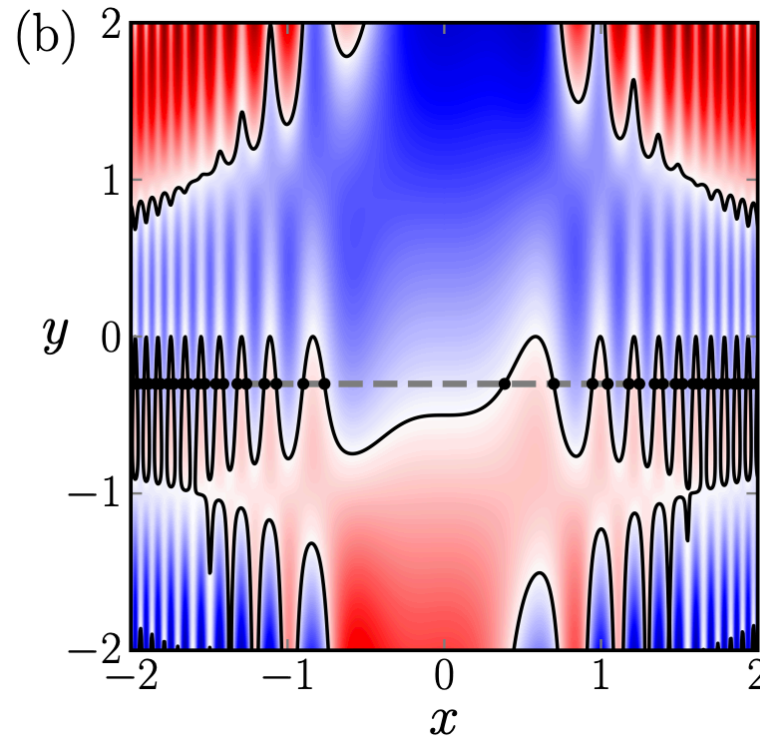
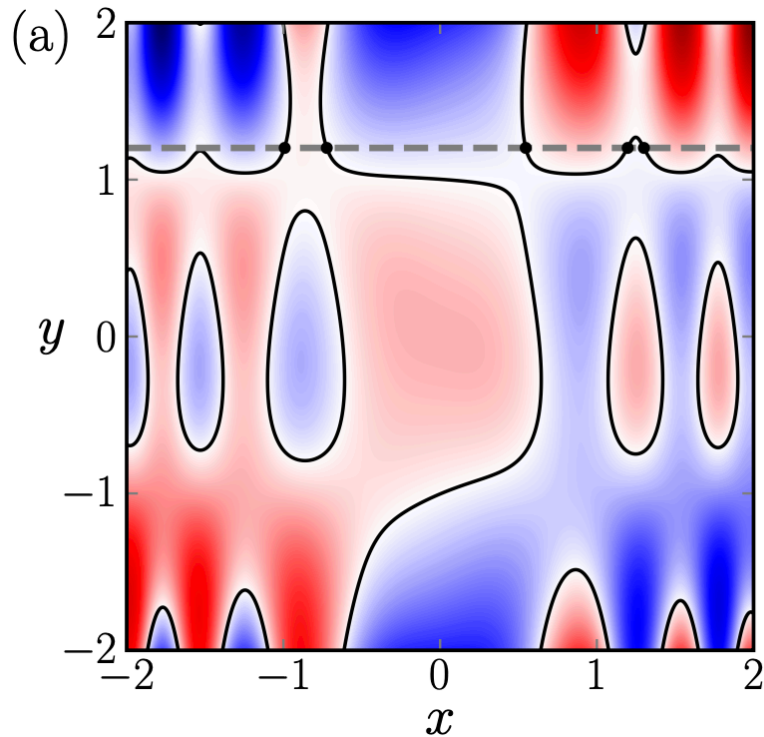
[Bogachev, 1998]



Zero-level sets of analytic functions

Infinite dimensions [B., Halikias, Otto, Townsend 2026]

If $\phi : \mathcal{F} \rightarrow \mathbb{R}$ is non-zero and **analytic** then $\phi^{-1}\{0\}$ is **compactly slice-null**



Zero-one law for one-shot recovery

Theorem. [B., Halikias, Otto, Townsend 2026]

Exactly one of the following is true:

- If f is sampled from a Gaussian measure on $\mathcal{F}$ then $f \notin \mathcal{F}_{\text{deg}}(\mathbf{c}^*)$ with probability 1 (recovery is almost-surely possible)
- All inputs are degenerate: $\mathcal{F}_{\text{deg}}(\mathbf{c}^*) = \mathcal{F}$ (recovery is impossible)

Proof. Suppose $f_0 \notin \mathcal{F}_{\text{deg}}(\mathbf{c}^*)$ and construct a non-zero analytic function

$\phi : \mathcal{F} \rightarrow \mathbb{R}$ such that $\mathcal{F}_{\text{deg}}(\mathbf{c}^*) \subset \phi^{-1}\{0\}$

compactly slice-null

Zero-one law for one-shot recovery

Suppose $f_0 \notin \mathcal{F}_{\text{deg}}(\mathbf{c}^*)$ and construct a **non-zero analytic function** $\phi : \mathcal{F} \rightarrow \mathbb{R}$ such that $\mathcal{F}_{\text{deg}}(\mathbf{c}^*) \subset \phi^{-1}\{0\}$

$L_{f_0, \mathcal{G}(f_0)} : \mathbf{c} \mapsto \sum_{i=1}^N c_i \mathcal{D}_i(\mathcal{G}(f_0), f_0)$ has rank N so we can find a basis

$\psi_1, \dots, \psi_N$ for the dual space of $W = \text{Range}(L_{f_0, \mathcal{G}(f_0)}) \subset \mathcal{V}$

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$$\phi : f \mapsto \det \begin{bmatrix} \psi_1(\mathcal{D}_1(\mathcal{G}(f), f)) & \cdots & \psi_1(\mathcal{D}_N(\mathcal{G}(f), f)) \\ \vdots & \ddots & \vdots \\ \psi_N(\mathcal{D}_1(\mathcal{G}(f), f)) & \cdots & \psi_N(\mathcal{D}_N(\mathcal{G}(f), f)) \end{bmatrix}$$

Zero-one law for one-shot recovery

Suppose $f_0 \notin \mathcal{F}_{\text{deg}}(\mathbf{c}^*)$ and construct a **non-zero analytic function** $\phi : \mathcal{F} \rightarrow \mathbb{R}$ such that $\mathcal{F}_{\text{deg}}(\mathbf{c}^*) \subset \phi^{-1}\{0\}$

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If $\mathcal{D}_i(\mathcal{G}(f), f)$ are **linearly dependent** then $\phi(f) = 0$, hence $\mathcal{F}_{\text{deg}}(\mathbf{c}^*) \subset \phi^{-1}\{0\}$

Zero-one law for one-shot recovery

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Proof. Suppose $f_0 \notin \mathcal{F}_{\text{deg}}(\mathbf{c}^*)$ and construct a non-zero analytic function

$\phi : \mathcal{F} \rightarrow \mathbb{R}$ such that $\mathcal{F}_{\text{deg}}(\mathbf{c}^*) \subset \phi^{-1}\{0\}$

 compactly slice-null

One-shot identification algorithm

Algorithm 2.1 Weak Identification of Analytic Systems (WIAS)

Require: Source term f , observed solution u , dictionary $\{\mathcal{D}_i\}_{i=0}^N$, dual space family

$$\{\psi_i\}_{i=1}^{r_0} \subset \mathcal{V}^*$$

- 1: Compute the $r_0 \times N$ matrix $G_{i,j} = \psi_i(\mathcal{D}_j(u, f))$
 - 2: Compute the right-hand side $b_i = \psi_i(\mathcal{D}_0(u, f))$, $1 \leq i \leq r_0$
 - 3: Compute a column-pivoted QR factorization $GP = QR$
 - 4: Determine the numerical rank r from the diagonal of R
 - 5: **if** $r < N$ **then**
 - 6: **return** non-identifiability from this input and test set
 - 7: Solve $R_{1:N,1:N} \mathbf{z} = Q_{:,1:N}^\top \mathbf{b}$
 - 8: Recover the coefficient vector by undoing the permutation: $\mathbf{c} = P\mathbf{z}$
 - 9: **return** $\mathbf{c}$
-

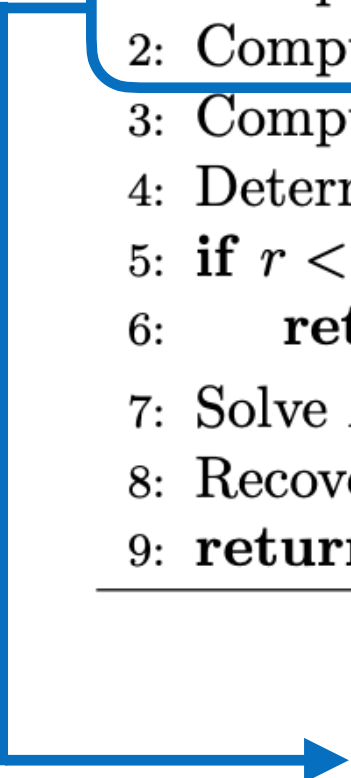
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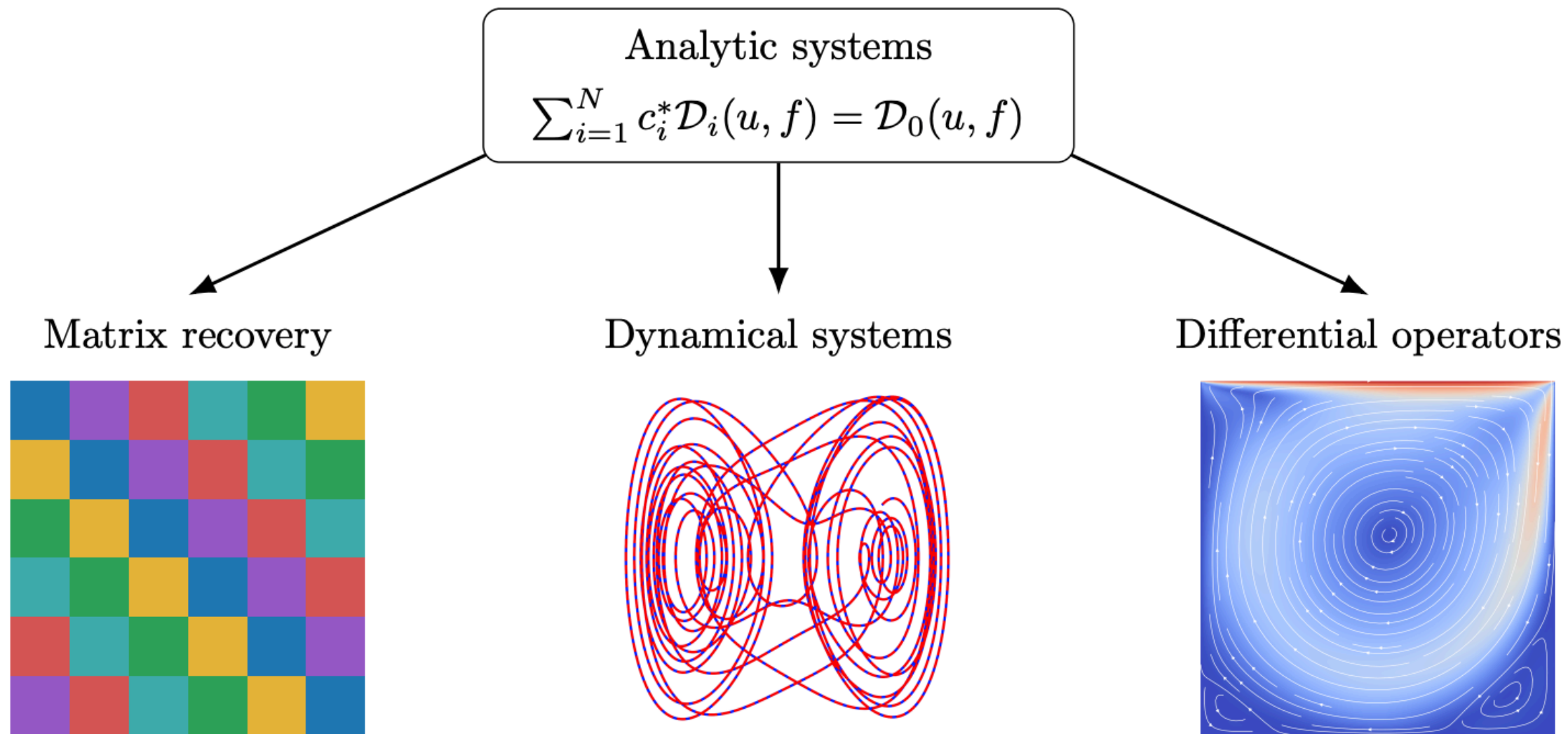
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 - 5: **if** $r < N$ **then**
 - 6: **return** non-identifiability from this input and test set
 - 7: Solve $R_{1:N,1:N} \mathbf{z} = Q_{:,1:N}^\top \mathbf{b}$
 - 8: Recover the coefficient vector by undoing the permutation: $\mathbf{c} = P\mathbf{z}$
 - 9: **return** $\mathbf{c}$
-

*Identify redundant
dictionary terms*


$$\begin{bmatrix} \psi_1(\mathcal{D}_1(u, f)) & \cdots & \psi_1(\mathcal{D}_N(u, f)) \\ \vdots & \ddots & \vdots \\ \psi_N(\mathcal{D}_1(u, f)) & \cdots & \psi_N(\mathcal{D}_N(u, f)) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} \psi_1(\mathcal{D}_0(u, f)) \\ \vdots \\ \psi_N(\mathcal{D}_0(u, f)) \end{bmatrix}$$

*Weak-SINDy
system $G\mathbf{c} = \mathbf{b}$*

Analytic systems identification



Examples of analytic systems identification

1. Matrix recovery

One seeks a linearly parametrized matrix

$$u = Af = \sum_{i=1}^N c_i^* A_i f$$

Corollary. [B., Halikias, Otto, Townsend 2026]

Sketch A and A^* with p and q Gaussian vectors, then **exactly** one of the following is true:

- Recovery is almost-surely possible
- All inputs are degenerate (recovery is impossible)

Structure	$\#x \mapsto Ax$	$\#x \mapsto A^\top x$
diagonal	1	-
block- k diagonal	k	0
tridiagonal	3	0
symm. tridiag.	2	-
rank- k	$k + p$	k
svmm. rank- k	$k + p$	-
circulant	1	0
Toeplitz or Hankel	2	0
symmetric	N	-
orthogonal	N	0
Toeplitz-like	$2p + 4$	$2p + 4$

[Halikias & Townsend 2024]

Examples of analytic systems identification

2. Dynamical systems

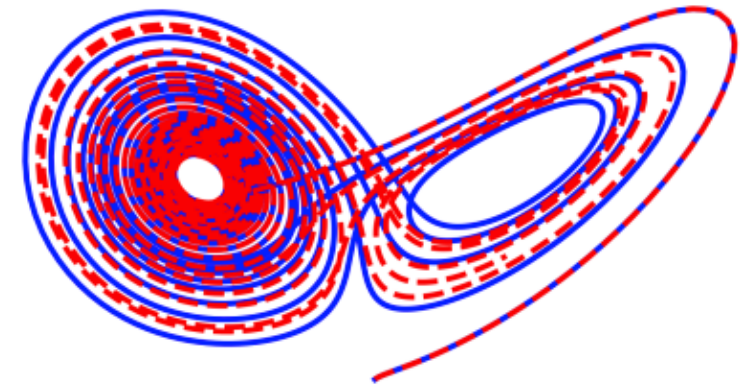
One seeks an ordinary differential equation

$$\dot{u}(t) = \sum_{i=1}^N c_i^* D_i(t, u, f)$$
$$u(0) = f_0$$

Corollary. [B., Halikias, Otto, Townsend 2026]

Exactly one of the following is true:

- If (f_0, f) is sampled from a Gaussian measure then recovery is almost-surely possible
- All inputs are degenerate (recovery is impossible)



Examples of analytic systems identification

3. Boundary value problems

$$\sum_{i=1}^N c_i^* \mathcal{D}_i(u) = f$$

One seeks a **partial differential equation**

If the dictionary functions depends on x and partial derivatives of u (e.g., x^2 , $u_x u_{yy}$)

Corollary. [B., Halikias, Otto, Townsend 2026]

If f is sampled from a Gaussian measure then **recovery is almost-surely possible**

In this case, one can construct a non-degenerate input f_0

A posteriori certification

Theorem. [B., Halikias, Otto, Townsend 2026]

Exactly one of the following is true:

- If f is sampled from a Gaussian measure on $\mathcal{F}$ then recovery is almost-surely possible
- All inputs are degenerate (recovery is impossible)

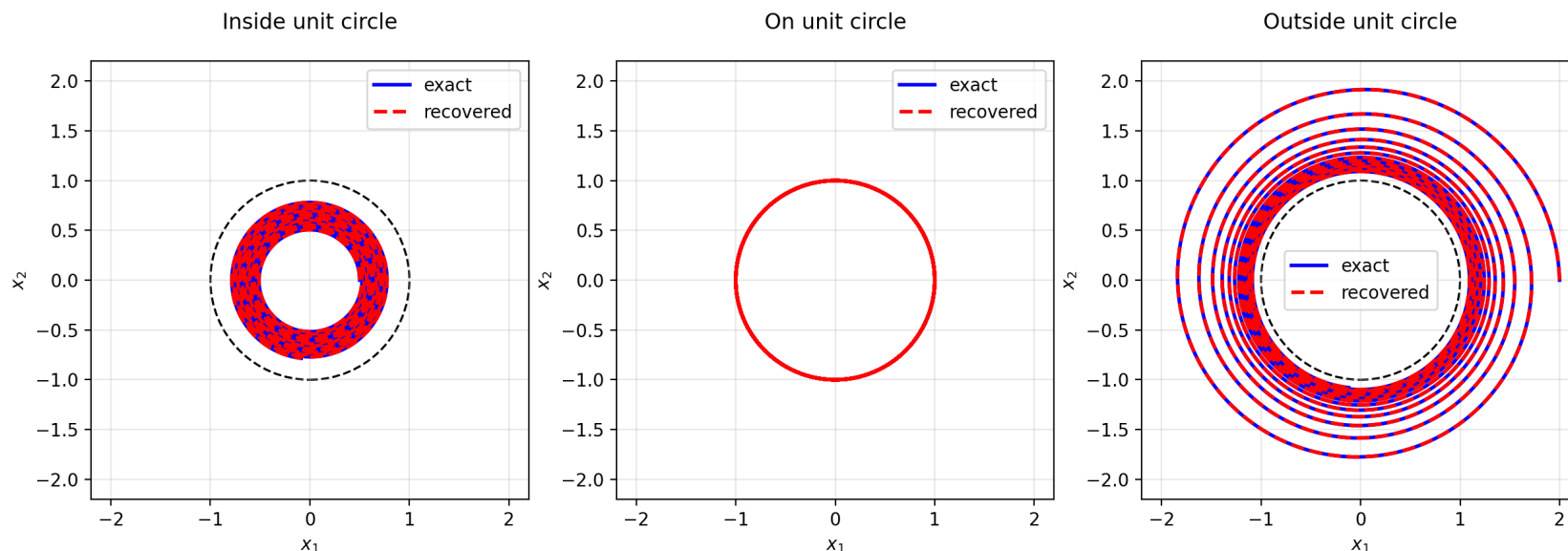
- Observe the response u to a random Gaussian input f
- Evaluate the dictionary terms $\mathcal{D}_i(u, f)$ through test functionals $\psi_1, \dots, \psi_N : \mathcal{V} \rightarrow \mathbb{R}$
- If the determinant of $(\psi_i(\mathcal{D}_j(u, f)))_{i,j}$ is non-zero then recovery is almost-surely possible

Numerical example: Hopf normal form

$$\dot{x}_1 = -x_2 - \alpha(r^2 - 1)x_1$$

$$\dot{x}_2 = x_1 - \alpha(r^2 - 1)x_2$$

Recovery system



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 1 & x & y & z & x^2 & xy & xz & y^2 & \dots & z^5 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

$\dot{\mathbf{X}} = \boldsymbol{\Theta}(\mathbf{X}) \boldsymbol{\Xi}$

Numerical example: Hopf normal form

$$\dot{x}_1 = -x_2 - \alpha(r^2 - 1)x_1$$

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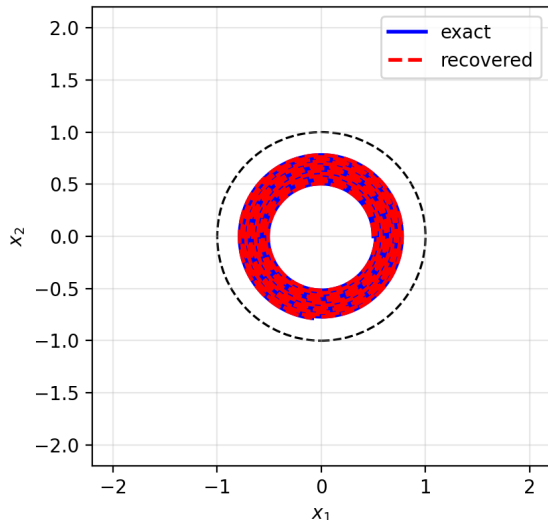
Recovery system



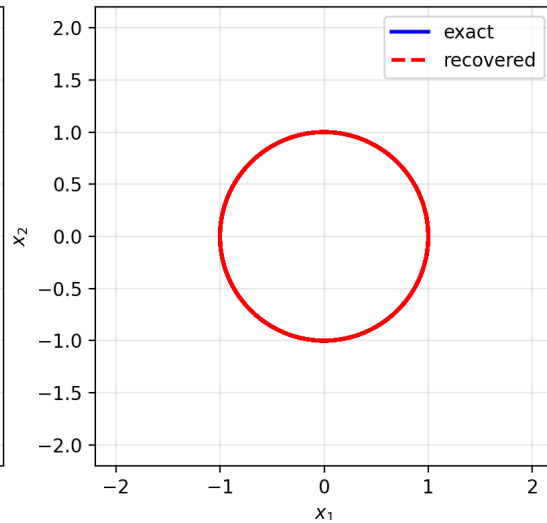
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 1 & x & y & z & x^2 & xy & xz & y^2 & \dots & z^5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

$\dot{X} = \Theta(X) \Xi$

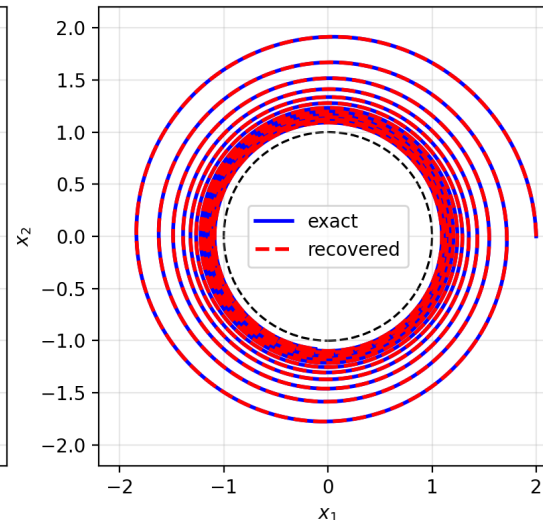
Inside unit circle



On unit circle

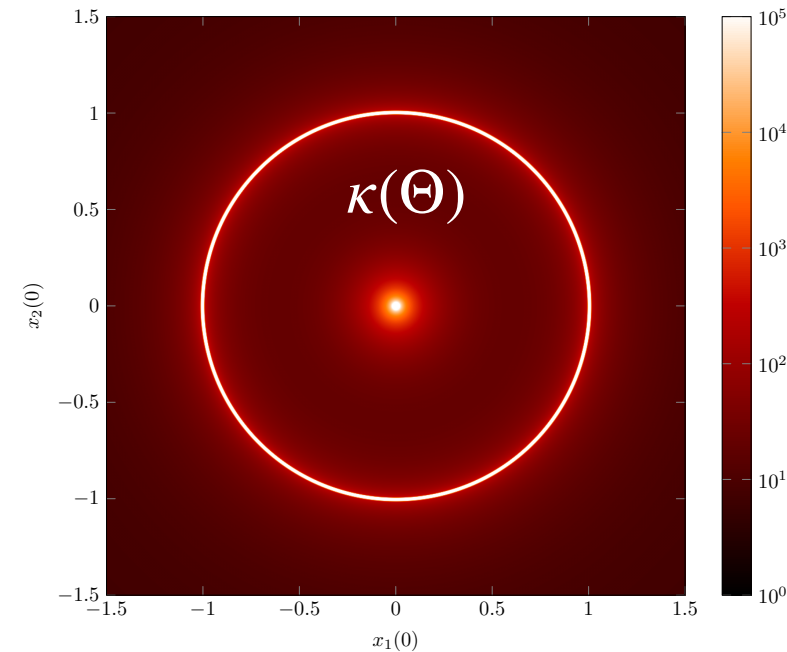


Outside unit circle



When $r = 1$, the model is **not unique** and the system is **ill-conditioned**:

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = -x_1$$

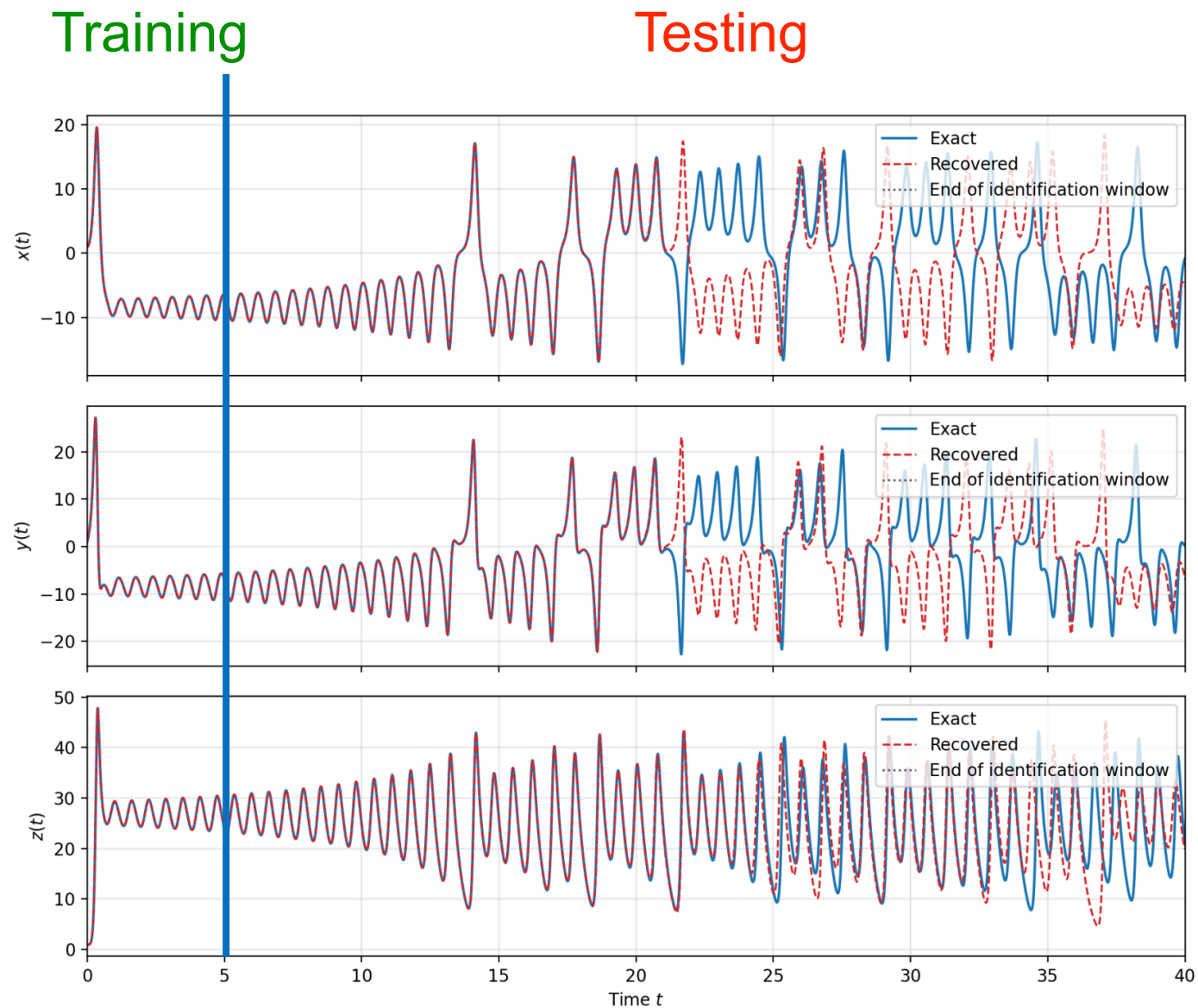
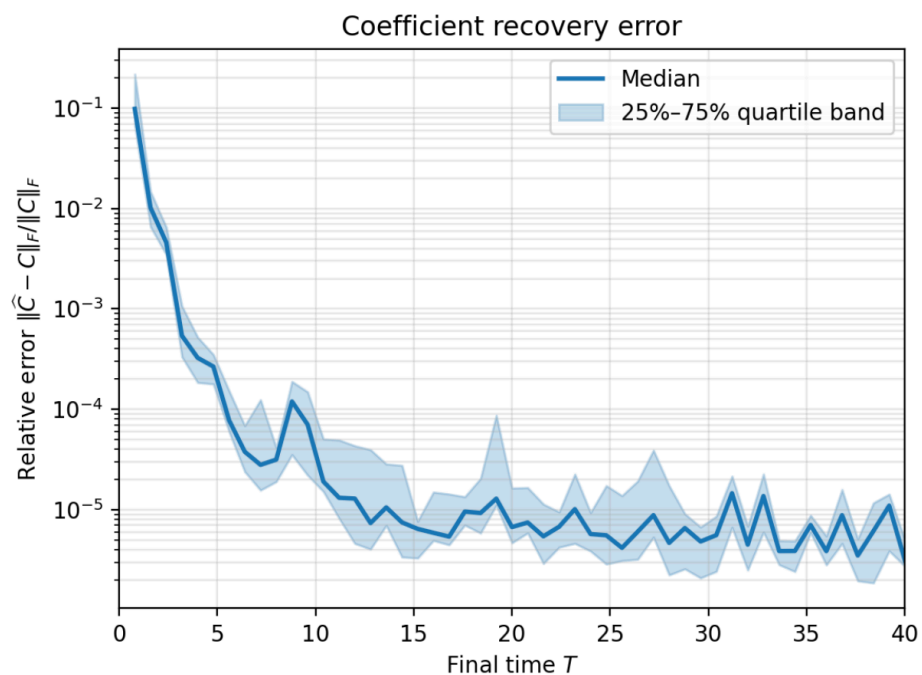


Numerical example: Lorenz system

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta z$$



Conclusions

1. Zero-one law for system identification

“One-shot recovery is either almost surely possible or never possible”

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2. Sample complexity guarantees for matrices, ODEs, PDEs

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“One-shot recovery is either almost surely possible or never possible”

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3. Explicit structure of degenerate input sets

